Approximation Schemes via Width/Weight Trade-offs on Minor-free Graphs

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Abstract

In this paper, we prove a new scaling lemma for vertex weighted minor free graphs that allows for a smooth trade-off between the weight of a vertex set S and the treewidth of $G - S$. More precisely, we show the following.

> There exists an algorithm that given an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and integers t and s, runs in polynomial time, and outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot \mathtt{opt}(G, w, t)/s$ such that the treewidth of $G - S$ is at most $c \cdot st$. Here, d and c are fixed constants that depend only on H, and $\mathtt{opt}(G, w, t)$ is the (unknown) minimum weight of a subset $U \subseteq V(G)$ such that the treewidth of $G - U$ is at most t.

This lemma immediately yields the first polynomial-time approximation schemes (PTASes) for Weighted Treewidth-η Vertex Deletion, for $\eta \geq 2$, on graphs of bounded genus and the first PTAS for Weighted Feedback vertex Set on H-minor free graphs. These results effortlessly generalize to include weighted edge deletion problems, to all Weighted Connected Planar $\mathcal{F}$-Deletion problems, and finally to quasi polynomial time approximation schemes (QPTASes) for all of these problems on H-minor free graphs. *For most of these problems even constant factor approximation algorithms, even on planar graphs, were not previously known.*

Additionally, using the scaling lemma we subsume, simplify and extend the recent framework of Cohen-Addad *et al.* [STOC 2016] for turning constant factor approximation algorithms for "ubiquitous" problems into PTASes for the same problems on graphs of bounded genus. Specifically, we obtain PTASes for ubiquitous problems without the requirement of having a constant factor approximation.

While the statement of the scaling lemma is inspired by an analogous lemma by Cohen-Addad *et al.* [STOC 2016] for edge contractions on weighted graphs of bounded genus, as well as a scaling lemma by Fomin *et al.* [SODA 2011] for unweighted graphs, the proof is entirely different. The proof detours via three different linear programming relaxations for the Weighted Treewidth-η Vertex Deletion problems and a strengthening of a recent rounding procedure of Bansal *et al.* [SODA 2017] enhanced by the classic Klein-Plotkin-Rao Theorem [STOC 1993].

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1 Introduction

In this paper, we prove a new scaling lemma (stated below) for vertex weighted minor free graphs that allows a smooth trade-off between the weight of a vertex set S and the treewidth of $G - S$.

Lemma 1.1. (Scaling Lemma) *There exists an algorithm that given an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$, and positive integers t and s, in polynomial time outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot \mathtt{opt}(G, w, t)/s$ such that $\mathtt{tw}(G - S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on H, $\mathtt{tw}(G - S)$ is the treewidth of $G - S$, and $\mathtt{opt}(G, w, t)$ is the minimum weight of a subset $U \subseteq V(G)$ such that $\mathtt{tw}(G - U) \leq t$.*

Lemma 1.1, combined with previously known results, enables to effortlessly obtain new approximation algorithms for a wide range of problems on H-minor free graphs and graphs of bounded genus. The class of problems that we consider are the Weighted Connected Planar $\mathcal{F}$-Deletion problems. Each family $\mathcal{F}$ of connected graphs, containing at least one planar graph, defines a Weighted Connected Planar $\mathcal{F}$-Deletion problem. Here, the input is a graph G and a weight function $w : V(G) \to \mathbb{Q}^+$. The goal is to find a minimum weight set S such that $G - S$ does not contain any of the graphs in $\mathcal{F}$ as a minor.[1] The Graph Minors Theorem [54] implies that, without loss of generality, the family $\mathcal{F}$ can be assumed to be finite. This definition captures problems such as Weighted Vertex Cover, Weighted Feedback Vertex Set, Weighted (Treewidth/Pathwidth/Treedepth)-η Deletion Set, Weighted Diamond Hitting Set and Weighted Outerplanar Vertex Deletion.

Weighted Connected Planar $\mathcal{F}$-Deletion problems (and their un-weighted counterparts) have received substantial attention both from the perspective of approximation algorithms [5, 7, 14, 29, 31, 39, 47] and parameterized algorithms [9, 16, 31, 56]. On planar and H-minor free graphs (the unweighted version of) Treewidth-η Vertex Deletion has played a central role in the developlemt of *Efficient* PTASes (EPTASes) via Bidimensionality [19, 23, 33, 34]. Nevertheless, prior to our work, most of these problems were not even known to admit constant factor approximation algorithms, even on planar graphs. The first collection of algorithmic consequences of Lemma 1.1 are encapsulated in the following theorem.

Theorem 1.1. (1) *Every* Weighted Connected Planar $\mathcal{F}$-Vertex Deletion *problem admits a PTAS on graphs of bounded genus.* (2) *Every* Weighted Connected Planar $\mathcal{F}$-Vertex Deletion *problem admits a $(1 + \epsilon)$-approximation algorithm on H-minor free graphs running in time $n^{\mathcal{O}(\log \log n \log \epsilon / \epsilon)}$.* (3) Weighted Feedback Vertex Set *admits a PTAS on H-minor free graphs.*

Here a *feedback vertex set* is a set S such that $G - S$ is acyclic, and in the Weighted Feedback Vertex Set problem the goal is to find a feedback vertex set of minimum weight.

To the best of our knowledge the only problems covered by Theorem 1.1 for which even constant factor approximation algorithms, even on planar graphs, were Feedback Vertex Set and Diamond Hitting Set (which admit constant factor approximation algorithms on general graphs [5, 29]), as well as "*local*" problems, where $\mathcal{F}$ contains at least one path, or equivalently, where every graph G that excludes all graphs in $\mathcal{F}$ as a minor have constant diameter. These problems are known to admit PTASes, in fact even EPTASes on minor-free graphs, by a classic result of Dawar et al. [17]. In the list above, Weighted Vertex Cover and Weighted Treedepth-η Vertex Deletion are local problems, while the others are not.

[1]The "connected" in the problem name refers to all graphs in $\mathcal{F}$ being connected. This is different from the meaning of "connected" in e.g. Connected Vertex Cover where the *solution set* S needs to induce a connected subgraph of G.

Problem Name	Planar graphs	Bounded genus	H-minor free
Wt Feedback Vertex Set	PTAS [14]	2 [5] / **PTAS**	2 [5] / **PTAS**
Wt Treewidth-η Deletion Set	$\mathcal{O}(\log^2 n)$ [1] / **PTAS**	$\mathcal{O}(\log^2 n)$ [1] / **PTAS**	$\mathcal{O}(\log^2 n)$ [1] / **QPTAS**
Wt Diamond Hitting Set	9 [29] / **PTAS**	9 [29] / **PTAS**	9 [29] / **QPTAS**
Wt "Local" $\mathcal{F}$-Deletion Problems	EPTAS [17]	EPTAS [17]	EPTAS [17]
Wt Connected $\mathcal{F}$-Deletion Problems	$\mathcal{O}(\log^2 n)$ [1] / **PTAS**	$\mathcal{O}(\log^2 n)$ [1] / **PTAS**	$\mathcal{O}(\log^2 n)$ [1] / **QPTAS**
Ubiquitous Problems	PTAS [14] / **PTAS***	PTAS [14] / **PTAS***	Unknown
Semi-ubiquitous Problems	**PTAS**	**PTAS**	Unknown

Table 1: Summary of the new results obtained in the paper (in **bold**) together with previously best known ratio and relevant citation. Here, Wt represents Weighted. Ubiquitous problems include e.g. Wt Connected Dominating Set and Tour Cover. Semi-ubiquitous problems (that are not Ubiquitous) include Wt Dominating Set with at most c-Components and Wt Cover by c-Tours. **PTAS*** means that the same results are obtained without needing a constant factor approximation for the considered problem as a prerequisite. The results of [1, 5, 17, 29] hold for classes more general than minor free.

Additionally, in a landmark result in 2016, Cohen-Addad et al. [14] obtained a PTAS for Weighted Feedback Vertex Set on planar graphs. The work of Cohen-Addad et al. [14] forms the starting point of this article. In particular, the *statement* of Lemma 1.1 is inspired by the main technical result of Cohen-Addad et al. [14] (restated here as Proposition 1.2) and a Scaling Lemma of Fomin et al. [33] for unweighted H-minor free graphs. We note that despite the similarity of statements, the *proof* of Lemma 1.1 is entirely different. We will discuss the results of Cohen-Addad et al. [14], as well as how this work extends theirs, in Sections 1.1 and 1.2. First, however, we demonstrate how to use Lemma 1.1 in order to obtain the approximation schemes claimed in Theorem 1.1.

Proof of Theorem 1.1. We begin by giving a PTAS for Weighted Feedback Vertex Set on H-minor free graphs. It is well known that the treewidth of acyclic graphs is at most 1. Therefore, the algorithm of Lemma 1.1 will output in polynomial time a vertex set S of weight at most $\epsilon \cdot OPT$, where OPT is the minimum weight of a feedback vertex set in G, such that the treewidth of $G - S$ is at most $\mathcal{O}(\log n/\epsilon)$. Using the $2^{\mathcal{O}(\mathtt{tw}(G))} n^{\mathcal{O}(1)}$ time algorithm of Cygan et al. [11] for Weighted Feedback Vertex Set, we can obtain a minimum weight feedback vertex set S' of $G - S$ in time $n^{\mathcal{O}(1/\epsilon)}$. Clearly, the weight of S' is at most OPT, and therefore $S \cup S'$ is a $(1+\epsilon)$-approximate solution to G.

To adapt this approach to work for all Weighted Connected Planar $\mathcal{F}$-Deletion problems we proceed as follows. First, for every planar graph H there exists a constant $t = \mathcal{O}(|V(H)|^{10})$ such that every graph G that excludes H as a minor has treewidth at most t [13]. Since $\mathcal{F}$ contains a planar graph, it follows that any solution U to Weighted Connected Planar $\mathcal{F}$-Vertex Deletion satisfies $\mathtt{tw}(G - U) \le t$. Thus, using Lemma 1.1 we obtain a vertex set S of weight at most $\epsilon \cdot OPT$ such that the treewidth of $G - S$ is at most $\mathcal{O}(t \log n/\epsilon)$. Here, OPT is the weight of the optimal solution U. Using the $2^{\mathcal{O}(\mathtt{tw}(G))} n^{\mathcal{O}(1)}$ algorithm of Baste et al. [8, 9] for Weighted Connected Planar $\mathcal{F}$-Vertex Deletion problems on graphs of bounded genus, we obtain an optimal solution S' of $G - S$ in time $n^{\mathcal{O}(t/\epsilon)}$. Clearly the weight of S' is at most OPT, and thus $S \cup S'$ is a $(1+\epsilon)$-approximate solution to G. Hence all Weighted Connected Planar $\mathcal{F}$-Deletion problems admit PTASes on graphs of bounded genus.

The only essential difference between the arguments for Weighted Feedback Vertex Set and for Weighted Connected Planar $\mathcal{F}$-Vertex Deletion is that the former has an algorithm with running time $2^{\mathcal{O}(\mathtt{tw}(G))} n^{\mathcal{O}(1)}$ on general graphs, whereas the latter only has an algorithm with such a running time on graphs of bounded genus. If we use the $2^{\mathcal{O}(\mathtt{tw}(G) \log \mathtt{tw}(G))} n^{\mathcal{O}(1)}$ algorithm of Baste et al. [8, 9] for Weighted Connected Planar $\mathcal{F}$-Vertex Deletion on *general* graphs, we obtain a $(1+\epsilon)$-approximation algorithm on H-minor free graphs running in time $n^{\mathcal{O}(\log \log n \log \epsilon/\epsilon)}$. This concludes the proof of Theorem 1.1. □

We remark that the $(1+\epsilon)$-approximation algorithms on H-minor free graphs running in time $n^{\mathcal{O}(\log \log n \log \epsilon / \epsilon)}$ are not PTASes, but they "over-qualify" as Quasi-Polynomial Time Approximation Schemes (QPTASes), since QPTASes are allowed to have polylogarithmic factors in n in the exponent of n.

1.1 Related Work

The study of PTASes on planar graphs is a popular research area whose history spans more than forty years [48]. In the early years, researchers primarily considered "local" graph problems on planar graphs [6]. The powerful layering technique of Baker [6, 40] is able to handle both weighted and un-weighted local problems on planar graphs with equal ease. For this reason the research focus shifted to "non-local" problems, such as Travelling Salesman [4, 37] and Feedback Vertex Set [46]. Approximation schemes for problems on planar graphs were designed on a problem to problem basis [37, 46], and weighted problems appeared to be more difficult than their unweighted counterparts—for example, it took 3 years to lift the PTAS for Travelling Salesman from unweighted to weighted planar graphs [4, 37], and 18 years to do the same for Feedback Vertex Set [46, 14].

Over time, the majority of known PTASes for both local and non-local problems on unweighted graphs were consolidated in the theory of bidimensionality [18, 19, 33], which in one broad stroke gave PTASes (even EPTASes) for a large class of problems on planar graphs. An advantage of bidimensionality is that it is "combinatorial"—it does not need an embedding of the input graph into a surface. For this reason, the approximation schemes obtained using bidimensionality work on classes substantially more general than planar graphs, such as H-minor free or apex-minor free classes of graphs, and even other geometric graph classes [19, 33].

At the same time, the picture for weighted and non-local problems on planar graphs is much more complex. Despite intensive study [4, 6, 17, 19, 25, 26, 35, 38, 42, 44, 48] and powerful design techniques, such as Klein's sparsification technique [43, 44] and contraction decomposition theorems [21, 22, 26], approximation schemes continue to be designed on a problem to problem basis and are mostly confined to planar or surface-embedded graphs. For example, basic problems such as Weighted Connected Dominating Set and Weighted Feedback Vertex Set did not have a PTAS on planar graphs until 2016 [14]. The approximability of Weighted Connected r-Dominating Set on planar graphs remains open.

In a recent important advance, Cohen-Addad et al. [14] introduced a new framework for designing PTASes for a class of *non-local* problems on *weighted* planar graphs and on graphs of bounded genus. Their framework allows to simultaneously obtain PTASes for Weighted Connected Dominating Set, Weighted Feedback Vertex Set, and many other problems on graphs of bounded genus. Nevertheless, we are still far from the same kind of understanding of the approximability of weighted problems on planar graphs, as we have of unweighted ones.

First, many important problems such as Weighted Treewidth-η Deletion Set problems for $\eta \geq 2$ are not captured by the framework of Cohen-Addad et al. [14]. Here, it is important to note that Treewidth-η Deletion Set problems play a *central* role in bidimensionality [18, 19, 33]. It is therefore quite natural to believe that a thorough understanding of Weighted Treewidth-η Deletion Set problems is crucial for building a general framework for approximation schemes on planar graphs and beyond. Second, the approach of Cohen-Addad et al. [14] is based on surface embeddings, and therefore "land-locked" to graphs of bounded genus. Indeed, their PTAS for Feedback Vertex Set relies on a clever duality trick, and therefore only works on planar graphs. It seems quite difficult to extend the results obtained by their approach to more general classes of graphs. Finally, the approximation schemes obtained by Cohen-Addad et al. [14] are all PTASes, while the unweighted versions of the considered problems all admit EPTASes. It is tempting to hypothesize that most of the problems that admit EPTASes on unweighted minor-free graphs also do on weighted minor-free graphs, and the paper of Cohen-Addad et al. [14], while a major advance, is a far cry from proving this.

In this paper, we substantially generalize the framework of Cohen-Addad et al. [14]. We address both the first and second issues simultaneously: our framework applies to a much larger class of problems (including WEIGHTED TREEWIDTH-η DELETION SET), and for many problems it works on the same classes of graphs as those covered by Bidimensionality (albeit at the cost of giving QPTASes instead of PTASes). Additionally, we come quite close to addressing the third issue—if the $\mathcal{O}(\log n)$ gap rounding algorithm in Section 7 can be improved to $o(\log n)$ or $o(\log n/\log\log n)$ then most of the PTASes and QPTASes automatically become EPTASes. If the gap becomes constant, they all do. We also remark that the framework of Cohen-Addad et al. [14] requires to have a constant factor approximation algorithm at hand (which may not be known a-priori), while our framework does not. A summary of the algorithmic results in our paper compared to previously known approximation algorithms can be found in Table 1.

Ubiquitous problems. Cohen-Addad et al. [14] consider problems that they termed *ubiquitous*: A problem is *t-ubiquitous* if, for every input graph G and every feasible solution S, *(i)* the graph $G[S]$ is connected, and *(ii)* the graph G/S (the graph obtained by contracting the edges of $G[S]$) has treewidth at most t.

We say that a minimization problem is *contraction closed* if contracting an edge and setting the weight of the resulting vertex to 0 can only decrease the value of the optimum. Cohen-Addad et al. [14] proved that if a minimization problem Π is t-ubiquitous, contraction closed and satisfies three more properties, then it admits a PTAS on weighted graphs of bounded genus. To properly compare their results with ours we re-state their main theorem here.

Proposition 1.1 ([14])**.** *Let Π be a contraction-closed t-ubiquitous minimization problem such that*

1. *Π admits a constant-factor approximation algorithm on graphs of bounded genus,*
2. *Π admits an exact (or $(1+\varepsilon)$-approximation) $2^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$-time algorithm on bounded genus graphs of treewidth at most k, and*
3. *Π has a* lifting algorithm*: There exits a constant β and a polynomial time algorithm that, given a graph G, vertex set S, and a solution X to problem Π for input G/S, outputs a solution for G of weight at most $w(S)+\beta\cdot w(K)$.*

Then, Π admits a PTAS on weighted graphs of bounded genus.

By direct application of Proposition 1.1, Cohen-Addad et al. derived the first PTASes for a number of fundamental "non-local" problems on weighted graphs of bounded genus, including the (edge-weighted) WEIGHTED TREE COVER and WEIGHTED TOUR COVER problems, and the (vertex-weighted) WEIGHTED CONNECTED DOMINATING SET, and WEIGHTED CONNECTED VERTEX COVER problems.

Theorem 1.1 and Proposition 1.1 apply to disjoint sets of problems because WEIGHTED CONNECTED PLANAR $\mathcal{F}$-VERTEX DELETION problems are not t-ubiquitous, and t-ubiquitous problems are not WEIGHTED CONNECTED PLANAR $\mathcal{F}$-VERTEX DELETION problems. However, using clever observations Cohen-Addad et al. [14] extended their results to give PTASes for two problems that do not directly fall within the scope of Proposition 1.1, namely for MAX-WEIGHT-LEAF SPANNING TREE on bounded genus graphs and for WEIGHTED FEEDBACK VERTEX SET on planar graphs. For comparison, the PTAS of Theorem 1.1 for WEIGHTED FEEDBACK VERTEX SET works on all H-minor free classes of graphs.

The main technical contribution of Cohen-Addad et al. [14] is a the following scaling lemma for *contraction* on *bounded genus* graphs.

Proposition 1.2 ([14])**.** *There exists an algorithm that given a graph G of genus g, a weight function $w : V(G) \to \mathbb{Q}^+$, a vertex set U such that $G[U]$ is connected, and an integer $s > 0$, in polynomial time outputs a subset $S \subseteq V(G)$ of weight at most $\mathcal{O}(\log n \cdot w(U)/s)$ such that $\mathtt{tw}(G/S) = \mathcal{O}(g^{\mathcal{O}(1)} \cdot s \cdot \mathtt{tw}(G/U))$.*

We remark that Cohen-Addad et al. [14] state Proposition 1.2 in a slightly different way, in particular they have weights both on vertices and on edges. The two different ways of stating Proposition 1.2 can easily be shown to be equivalent. The proof of Proposition 1.2 given Proposition 1.1 goes along the same lines as the proof of Theorem 1.1 given Lemma 1.1 (or rather, it is the other way around).

1.2 PTASes for Ubiquitous Problems: Simplified and Generalized

Starting from Lemma 1.1 we are able to obtain the following strengthening of Proposition 1.2. We will discuss how to derive Lemma 1.2 from Lemma 1.1 in Section 1.3.

Lemma 1.2. *For every pair of integers $t \geq 1$, $g \geq 0$, there exists an algorithm that given a graph G of genus g, a weight function $w : V(G) \to \mathbb{Q}^+$, and an integer s, in polynomial time outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot \mathtt{opt}_{VC}(G, w, t)/s$ such that $\mathtt{tw}(G/S) \leq c \cdot s$. Here, d and c are fixed constants that depend only on g and t, and $\mathtt{opt}_{VC}(G, w, t)$ is the minimum weight of a subset $U \subseteq E(G)$ such that $\mathtt{tw}(G/U) \leq t$. The degree of the polynomial bounding the running time is independent of t and g.*

Lemma 1.2 is a strengthening of Proposition 1.2 in the sense that Proposition 1.2 requires the set U to be connected, and to be given as input, while Lemma 1.2 does not. This comes at the price that Lemma 1.2 has additional factors that are exponential in t in the running time of the algorithm and the bounds on the weight and treewidth of G/S, while Proposition 1.2 does not. This overhead does not matter for our applications, since we always consider both g and t to be constants.

By replacing Proposition 1.2 with Lemma 1.2 in the arguments of Cohen-Addad et al. [14], we obtain a substantial strengthening of their main result. We start by defining the class of problems that our results apply to, and show that they are a strict generalization of the t-ubiquitous problems considered by Cohen-Addad et al. [14].

Definition 1.1. (Semiubiquitous Problems) *A problem P on (vertex-weighted) graphs is t-*semiubiquitous *if there exists a constant c such that for every input graph G and weight function $w : V(G) \to \mathbb{Q}^+$, there exists a set $S \subseteq V(G)$ of weight at most $c \cdot \mathtt{opt}_P(G, w, t)$ such that G/S has treewidth at most t.*

Observe that if a problem P is t-ubiquitous then it is also t-semiubiquitous, because every optimal solution S to a t-ubiquitous problem P satisfies the conditions of the set S in Definition 1.1. On the other hand, not all semiubiquitous problems are ubiquitous: Definition 1.1 does not require feasible solutions to be connected, and does not require every feasible solution X to satisfy $\mathtt{tw}(G/X) \leq t$. Instead, we only need a set S of weight comparable to that of the optimal solution, such that $\mathtt{tw}(G/S) \leq t$. We are now in position to state and prove our strengthening of Proposition 1.1.

Theorem 1.2. *Let Π be a contraction closed, t-semiubiquitous minimization problem such that*

1. *Π admits an exact (or $(1+\varepsilon)$-approximation) $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$-time algorithm on bounded genus graphs of treewidth at most k, and*
2. *Π has a* lifting algorithm*: There exits a constant β and a polynomial time algorithm that, given a graph G, vertex set S, and a solution X to problem Π for input G/S, outputs a solution for G of weight at most $w(X) + \beta \cdot w(S)$.*

Then, Π admits a PTAS on weighted graphs of bounded genus.

Proof. First, apply the algorithm of Lemma 1.2 and obtain a set S of weight $\delta \cdot OPT$ such that $\mathtt{tw}(G/S) \leq O(\log n/\epsilon)$. Then use the $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$-time algorithm to obtain a $(1+\alpha)$-approximate solution X to G/S. Since Π is contraction closed, $w(X) \leq (1+\alpha)OPT$ where OPT is the weight of an optimal solution in G. Finally, use the lifting procedure on X and S to get a solution to G of weight at most $(1+\epsilon)OPT$, where $\epsilon = \alpha + \delta \cdot \beta$. □

Since every semiubiquitous problem is also ubiquitous, Theorem 1.2 applies to all problems that Proposition 1.1 applies to. Additionally, because it does not require a constant factor approximation, Theorem 1.2 is easier to apply. For example, to obtain a PTAS for Weighted Connected Dominating Set, Cohen-Addad et al. [14] first needed to design a constant factor approximation for the problem (on bounded genus graphs). Finally, Theorem 1.2 also yields PTASes for the variants of Weighted Vertex Cover, Weighted Connected Dominating Set, Tree Cover and Tour Cover where the solution does not have to be connecteded but instead have at most c connected components for a fixed constant c. Here we omit the details of verifying that these variants of the problems still satisfy the conditions of Theorem 1.2; they are essentially identical to the discussion for the original problems by Cohen-Addad et al. [14].

1.3 Extensions and Further Applications of the Scaling Lemma

At first glance the applicability of Lemma 1.1 seems limited to vertex subset problems. This turns out not to be the case. Indeed, with relative ease, we can extend Lemma 1.1 to variants of edge deletion and edge contraction.

Lemma 1.3. *There exists an algorithm that given an H-minor free graph G, a weight function $w : E(G) \to \mathbb{Q}^+$, and positive integers t and s, in polynomial time outputs a subset $S \subseteq E(G)$ of weight at most $d \log n \cdot \mathtt{opt}_E(G, w, t)/s$ such that $\mathtt{tw}(G - S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on H, and $\mathtt{opt}_E(G, w, t)$ is the minimum weight of a subset $U \subseteq E(G)$ such that $\mathtt{tw}(G - U) \leq t$.*

Since it is easy to derive Lemma 1.3 from Lemma 1.1, we give a proof sketch here. A full proof of Lemma 1.3 is given in Section 8.

Proof sketch. To derive Lemma 1.3 from Lemma 1.1, it suffices to transform the edge weighted input instance (G, w) to a vertex weighted instance (G', w') where G' is equal to G with all edges subdivided, and w' assigns infinite weight to vertices of G' that correspond to vertices of G, and for every vertex $v_e \in V(G')$ that corresponds to an edge $e \in E(G)$, we set $w'(x_e) = w(e)$. Applying Lemma 1.1 to G' and w', and translating the output set $S \subseteq V(G')$ back to an edge set of G, completes the proof. □

Lemma 1.3 implies precisely the same conclusions (PTASes on bounded genus graphs, QPTASes on H-minor free graphs) for the edge deletion versions of Weighted Connected Planar $\mathcal{F}$-Vertex Deletion problems. (In the edge deletion versions, the weights are on edges and we seek a minimum weight edge set S such that $G - S$ excludes all graphs in $\mathcal{F}$ as a minor.) The results follow because the algorithms of Baste et al. [8, 9] extend also to these edge deletion variants. We remark that Baste et al. [8, 9] do not state this explicitly, but their arguments for vertex deletion apply (almost) without modification to edge deletion problems. For ease of reference we state our results for Weighted Connected Planar $\mathcal{F}$-Edge Deletion in a separate theorem. Note that a PTAS for the edge deletion version of Weighted Feedback Vertex Set is meaningless, as this problem is equivalent to Maximum Weight Spanning Tree, and therefore solvable in polynomial time.

Theorem 1.3. (1) *Every* Weighted Connected Planar $\mathcal{F}$-Edge Deletion *problem admits a PTAS on graphs of bounded genus.* (2) *Every* Weighted Connected Planar $\mathcal{F}$-Edge

Deletion *problem admits a $(1+\epsilon)$-approximation algorithm on H-minor free graphs running in time $n^{\mathcal{O}(\log\log n \log \epsilon/\epsilon)}$.*

In the rest of this section we will outline the proof of Lemma 1.2. For graphs of bounded genus it is easy to apply Lemma 1.3 and derive a scaling lemma for edge contraction.

Lemma 1.4. *There exists an algorithm that given a graph G of genus g, a weight function $w : V(E) \to \mathbb{Q}^+$, and positive integers t and s, in polynomial time outputs a subset $S \subseteq E(G)$ of weight at most $d \log n \cdot \mathtt{opt}_{EC}(G, w, t)/s$ such that $\mathtt{tw}(G/S) \le c \cdot st$. Here, d and c are fixed constants that depend only on g and $\mathtt{opt}_{EC}(G, w, t)$ is the minimum weight of a subset $U \subseteq E(G)$ such that $\mathtt{tw}(G/U) \le t$.*

Just as for Lemma 1.3, we give a proof sketch here and defer the full proof of Lemma 1.4 to Section 8.

Proof sketch. If the graph G can be embedded in a surface of genus g, then the treewidth of G and the treewidth of the dual graph G^* of G are equal, up to an additive term of $g+1$ (see Mazoit [50]). Further, there is a one-to-one correspondence between edges of G and G^*, and contracting edges in G is the same as deleting edges in the dual G^*. In particular, for any edge set $U \subseteq E(G)$, the dual of G/U is $G^* - U$. Thus we can apply Lemma 1.3 to the dual graph G^*, and the edge set S of G^* output by Lemma 1.3, when interpreted as an edge set of G, has the desired properties. Here, to upper bound the treewidth of G/S, we again use the result of Mazoit [50] that the treewidth of G/S and $G^* - S$ are equal up to an additive term of $g+1$. □

Lemma 1.4 does not yield any immediate algorithmic consequences for contraction variants of Weighted Connected Planar $\mathcal{F}$-Vertex Deletion problems, because the algorithms of Baste et al. [8, 9] do not apply to edge contraction problems. It is quite plausible that the techniques of Baste et al. [8, 9] can be adapted to yield algorithms with similar running times even for contraction problems, however this is pure speculation.

Lemma 1.4 looks conspicuously like Lemma 1.2. An important difference is that Lemma 1.4 deals with edge-weighted graphs, whereas Lemma 1.2 deals with vertex weighted graphs. It turns out that this gap can be bridged, and a contraction scaling lemma for *vertex* weighted graphs that *almost* matches the statement of Lemma 1.2 can be derived from Lemma 1.4.

Lemma 1.5. *There exists an algorithm that given a graph G of genus g, a weight function $w : V(G) \to \mathbb{Q}^+$, positive integers t and s, and a subset $U \subseteq V(G)$ such that $\mathtt{tw}(G/U) \le t$, in polynomial time outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot w(U)/s$ such that $\mathtt{tw}(G/S) \le c \cdot st$. Here, d and c are fixed constants that depend only on g.*

Deriving Lemma 1.5 from Lemma 1.4 requires a bit more effort than the previous steps, and so we shall not give a proof sketch here and rather point the reader to the full proof in Section 8.1. We remark that this proof is relatively simple (it spans a couple pages) and relies only on combinatorial arguments. An ingredient of this proof that may have independent interest is a purely combinatorial result that states that contracting a set of vertex disjoint stars in an apex-minor-free graph G can only decrease the treewidth of G by a constant factor (Lemma 8.2).

The reason Lemma 1.2 does not immediately follow form Lemma 1.5 is that Lemma 1.5 requires the set U to be given as input. On the other hand, in Lemma 1.2 the weight of the output set S is compared against the "best possible" U. To derive Lemma 1.2 from Lemma 1.5 we will use an approximation algorithm for the following problem.

Weighted Treewidth-η Vertex Contraction
Instance: A graph G, and a weight function $w : V(G) \to \mathbb{Q}^+$.
Objective: Find a minimum weight set $S \subseteq V(G)$ such that $\mathtt{tw}(G/S) \le \eta$.

Our approximation algorithm crucially uses that the output set S does not have to be connected. In particular, an approximation algorithm for Weighted Treewidth-η Vertex Contraction together with Proposition 1.2 instead of Lemma 1.5 could *almost* be used to prove Lemma 1.2, but this does not quite work out because Proposition 1.2 needs the set U (and therefore the output of the approximation algorithm) to be connected.

Theorem 1.4. *For every fixed constant $\eta \in \mathbb{N}$ and graph H there exists a constant factor approximation algorithm for* Weighted Treewidth-η Vertex Contraction *on H-minor-free graphs. The approximation ratio and the multiplicative constant of the running time of the algorithm depend on H and η, while the degree of the running time does not*[2].

The algorithm of Theorem 1.4, given in Section 9, works by combining the local-ratio technique with Myhill-Nerode equivalence style arguments, and is interesting in its own right. Combining Lemma 1.5 and Theorem 1.4, we obtain a scaling lemma for vertex contraction where the set U does not have to be given as input.

Proof of Lemma 1.2. The algorithm first calls the approximation algorithm of Theorem 1.4 and obtains in polynomial time a set U of weight at most $d' \cdot \mathtt{opt}_{VC}(G, w, t)$ such that $\mathtt{tw}(G/U) \leq t$. The value of d' depends only on t and g. Now the algorithm applies Lemma 1.5 on G, w, and U to obtain in polynomial time a set S of weight at most $d'' \log n \cdot w(U)/s \leq d'' \log n \cdot d' \cdot \mathtt{opt}_{VC}(G, w, t)$ such that $\mathtt{tw}(G/S) \leq c'' \cdot st$. Here, d'' and c'' depend only on the genus g of G. Setting $c = c't$ and $d = d' \cdot d''$, we observe that S satisfies the statement of the lemma. □

1.4 Brief Proof Outline for Lemma 1.1

Both the proof of Fomin et al. [33] of the unweighted scaling lemma and our proof of Lemma 1.1 builds on the linear relationship between the treewidth of an H-minor-free graph G and the size of the largest grid minor in G.

Let $gm(G)$ be the largest t such that G contains a $t \times t$ grid as a minor. It is easy see that $\mathtt{tw}(G) \geq gm(G)$. On the other hand, Demaine and Hajiaghayi showed [20] that for every graph H there exists a constant c_H such that $\mathtt{tw}(G) \leq c_H \cdot gm(G)$. Thus, for minor free graphs $gm(G)$ and $\mathtt{tw}(G)$ are essentialy the same. For ease of notation, in the rest of this outline we will treat them as if they *are* the same. Thus, a subset of vertices S satisfies $\mathtt{tw}(G - S) \leq t$ if and only if it intersects every $t \times t$ grid minor model in G.

Observe now that an $(rt) \times (rt)$ grid minor model contains r^2 disjoint $t \times t$ grid minor models. Thus, in any graph one can *pack* a factor of r^2 more disjoint $t \times t$ grid minor models as $(rt) \times (rt)$ grid minor models. With a leap of faith one can conclude that the same should hold for *hitting* grid minor models. In other words, that hitting all $t \times t$ grid minor models should require a factor of r^2 more vertices than hitting all $(rt) \times (rt)$ grid minor models. What we are really interested in is the contrapositive of this statement, that hitting all $(rt) \times (rt)$ grid minor models requires a factor of r^2 *fewer* vertices than hitting all $t \times t$ grid minor models. For *unweighted* minor free graphs this intuition can be turned into a relatively simple proof [33]. For weighted graphs we do not know how to do this in a direct way, and so we take an indirect route through the world of Linear Programming (LP).

Consider the natural LP relaxation of Weighted Treewidth-η Deletion Set where every vertex v gets a variable x_v that takes values between 0 and 1. Here 1 means that the vertex is included in the solution S and 0 means that it is not. Naturally we wish to minimize $\sum_{v \in V(G)} w_v x_v$ subject to the constraint that for every $t \times t$ grid minor model the sum of the variables in it should be at least 1.

[2] That is, the time complexity is of the form $f(H, \eta) \cdot n^s$ where s is independent of H and η.

It is easy to see that the scaling property we seek (even a stronger one, without the $\log n$ factor) holds for the this LP relaxation. In particular consider a feasible assignment x to the variables for hitting all $t \times t$ grid minor models. Observe now that x/r^2 is a feasible assignment to the variables for hitting all $tr \times tr$ grid minor models, and that this assignment is a factor of r^2 cheaper than the original x. This is precisely what we need - making the solution set cheaper at the cost of making the treewidth of the graph with the solution removed larger.

Observe now that the characteristic vector of a vertex set S such that $\texttt{tw}(G - S) \leq t$ is a feasible solution to the $t \times t$ grid hitting LP. Thus, by the arguement above we can make it a factor r^2 cheaper by fractionally hitting $rt \times rt$ grids instead. If we could round this LP solution with a gap of $o(r^2)$ we would obtain a cheaper set S' (of weight at most $w(S) \cdot \frac{o(r^2)}{r^2} \leq \epsilon w(S)$) such that $\texttt{tw}(G - S') \leq rt$. Unfortunately we do not know how to round solutions to the grid hitting LP.

Fortunately, Bansal et al. [7] recently introduced a different LP relaxation for the Weighted Treewidth-η Deletion Set problem (we will call this the *well-linkedness* LP) and proved that the (edge deletion) version of this LP can be rounded with gap $\mathcal{O}(\log n \log\log n)$, even on general graphs.

In order to use the well-linkedness LP in order to round solutions to the grid hitting LP we first need to make make a vertex version of the well-linkedness LP, and then establish a relationship between solutions to the grid hitting LP and solutions to the well-linkedness LP. On of the main technical contributions of this paper is to establish such a connection. In particular we show that an assignment x is feasible for $t \times t$ grid hitting LP if and only if $x \cdot t$ is a feasible solution for the well-linkedness LP relaxation of Weighted Treewidth-t Deletion Set. We remark that on general graphs this relationship does *not* hold.

We can now try to use the LP scaling above together with the rounding algorithm of Bansal et al. [7] (in fact an adaptation of their rounding algorithm to the vertex version, which is non-trivial!) to prove Lemma 1.1. This *almost* works. Start with S, this is a feasible solution to the well-linkedness LP. Using the "only if" direction of the relationship between LP relaxations we see that assigning $1/t$ to every vertex in S is a feasible solution to the $t \times t$ grid hitting LP. By scaling the grid hitting LP we see that assigning $1/r^2t$ to every vertex in S is a feasible solution to the $(rt) \times (rt)$ grid hitting LP. By using the "if" direction of the relationship between LPs we conclude that assigning $1/r$ to every vertex in S is a feasible solution to the well-linkedness LP for deletion to treewidth tr.

Rounding this solution using the method of Bansal et al. [7] leads to a set S' of weight at most $w(S)/r \cdot \log n \log\log n$ s.t. $\texttt{tw}(G - S') \leq tr$. To make $w(S') \leq \epsilon S$ we need to pick $r \leq \log n \log\log n$ implying that $\texttt{tw}(G - S') = O(\log n \log\log n)$. This is almost what we claim in Lemma 1.1, but it is off by a factor of $\log\log n$. This factor turns out to be a big problem for our applications, because treewidth based algorithms have running times that are at least exponential in the treewidth. Since $2^{\log n}$ is polynomial in n while $2^{\log n \log\log n}$ is quasipolynomial, this approach appears to yield QPTASes, but not PTASes.

We overcome this problem by designing our own rounding algorithm for (the vertex version of) the well-linkedness LP with gap $\mathcal{O}(\log n)$. Our rounding algorithm is essentialy the same as that of Bansal et al. [7], but it uses the Klein Plotkin Rao balanced separator rounding [45] for minor-free graphs, and needs needs some non-trivial modifications to work for the vertex version instead of the edge version.

2 Proof Outline of Lemma 1.1

In this section, we give a slightly more technical overview before proceeding with the formal proof. We do not define standard or self-explanatory notations here to keep this overview light (or rather, as light as possible!); formal definitions can be found in Section 3.

2.1 An LP-formulation that is useful for rounding

Recall the WEIGHTED TREEWIDTH-η VERTEX DELETION problem. Here, given a graph G and a weight function $w : V(G) \to \mathbb{Q}^+$, the objective is to find a set X of minimum weight such that $\mathtt{tw}(G - X) \leq \eta$. An LP for the edge version of this problem, called the BOUNDED TREEWIDTH INTERDICTION problem, was given by Bansal et al. [7]. Towards this, they used the notion of *well-linked sets*: a vertex set S is *t-linked* in G if S does not have a $\frac{1}{2}$-separator X in G with $|X| < t$. Moreover, the *linkedness* of G, denoted by $\mathtt{link}(G)$, is the maximum integer t such that there exists a t-linked set in G. It is known from [52] that $\mathtt{link}(G) < \mathtt{tw}(G) \leq 4\mathtt{link}(G)$. Bansal et al. [7] used this approximate characterization of treewidth to formulate the LP, and gave a bicriteria $(\log n \log\log n, \log \eta)$-approximation algorithm for the BOUNDED TREEWIDTH INTERDICTION problem on general graphs. That is, they gave an algorithm that runs in polynomial time and output a set F' such that $w(F') \leq \mathcal{O}(\log n \log\log n \cdot \mathtt{opt}(G, w, \eta))$ and $\mathtt{tw}(G - F') \leq \mathcal{O}(\eta \log \eta)$. Here, $\mathtt{opt}(G, w, \eta)$ is the (unknown) minimum weight of a subset $U \subseteq E(G)$ such that $\mathtt{tw}(G - U) \leq \eta$. We refer to the LP formulation used for this result as **Well-Linkedness LP**. Let us state the vertex version of this LP:

Well-Linkedness LP(G, w, t)**:**

$$\min \sum_{v \in V(G)} w_v x_v$$

$$\text{s.t.} \quad \sum_{v \in V(G)} y_v^S \leq t \qquad \forall S \subseteq V(G)$$

$$d_{uv}^S \leq \sum_{r \in V(P)} (x_r + y_r^S) \qquad \forall S \subseteq V(G), u, v \in S, P \in \mathcal{P}_G(u, v)$$

$$\sum_{v \in U} d_{uv}^S \geq |U| - \frac{|S|}{2} \qquad \forall U \subseteq S \subseteq V(G), u \in U$$

$$x_v \geq 0, y_v^S \geq 0, d_{uv}^S \geq 0$$

In this LP, we are given a graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and $t \in \mathbb{N}$. Our objective is to delete vertices of minimum weight so that G will not have a $(t+1)$-linked set. Each variable x_v indicates whether we delete v. For every subset $S \subseteq V(G)$, we have a variable y_v^S for all $v \in V(G)$, and a variable d_{uv}^S for all $u, v \in S$. Informally, these variables exhibit a $\frac{1}{2}$-separator Y of S in G with $|Y| \leq t$ as follows. First, each variable y_v^S indicates whether v belongs to Y. Now, each variable d_{uv}^S encodes the distance (lightest path) between u and v in the metric where the weight of each vertex r is given by $(x_r + y_r^S)$. In the constraints, we have that $d_{uv}^S \leq \sum_{r \in V(P)} (x_r + y_r^S)$ for all $P \in \mathcal{P}_G(u, v)$, and since it is "beneficial" too keep d_{uv}^S as large as possible, this means that it can be thought of as being equal to the distance $\sum_{r \in V(P)} (x_r + y_r^S)$ above. Now, to ensure that Y is a $\frac{1}{2}$-separator of S in G, it should (roughly) hold that in this metric, for every vertex $u \in S$ that can exist at most $\frac{|S|}{2}$ vertices at distance smaller than 1 from u. This is encoded by having a constraint $\sum_{v \in U} d_{uv}^S \geq |U| - \frac{|S|}{2}$ for all $U \subseteq S \subseteq V(G)$ and $u \in U$. Here, the consideration of every subset $U \subseteq S$ aims to replace the need to cap (bound from above) the distances d_{uv}^S by 1 and hence lose the property of encoding a metric. Specifically, for a vertex $u \in S$, our purpose is to satisfy the constraint where U is the set of all vertices at distance at most 1 from u. This LP has some very good properties that make it useful in our *rounding procedure* (see Section 2.4).

2.2 An LP-formulation that is useful for scaling

We also need an LP that has a good "scaling property". For this recall that our objective is to design a polynomial time algorithm that takes as input an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and integers t and s (which can depend on G), and outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot \texttt{opt}(G, w, t)/s$ such that $\texttt{tw}(G - S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on H and $\texttt{opt}(G, w, t)$ is the (unknown) minimum weight of a subset $U \subseteq V(G)$ such that $\texttt{tw}(G - U) \leq t$. It is well known if an H-minor free graph has treewidth at most t, then it excludes a $t' \times t'$ grid as a minor where $t' = \mathcal{O}(t)$ [23]. Thus, another natural LP that can be associated with our problem is the one that has a "hitting constraints" for every small (but not too small) subgrid in G. We refer to this formulation as **Grid Hitting LP**:

Grid Hitting LP(G, w, t)**:**

$$\begin{aligned}
\min \quad & \sum_{v \in V(G)} w_v x_v \\
\text{s.t.} \quad & \sum_{v \in S} x_v \geq 1 && \forall S \in \texttt{Grid}_t(G) \\
& x_v \geq 0
\end{aligned}$$

In **Grid Hitting LP**, we are given a graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and $t \in \mathbb{N}$. Our objective is to hit all $t \times t$-grids in G. Each variable x_v indicates whether we delete v, and the constraints are self-explanatory. The rationale behind the definition of this LP, in the context of our work, stems from the following known results. On the one hand, we have the following relation.

Proposition 2.1 ([23])**.** *Let H be a graph. There exists a fixed constant $c = c(H)$ such that for any H-minor free graph G of treewidth lower bounded by ct, it holds that G has a $t \times t$-grid as a minor.*

On the other hand, the following result implies that if G has a $t \times t$-grid as a minor, then its treewidth is lower bounded by t.

Proposition 2.2 ([15])**.** *The treewidth of a $t \times t$ grid is exactly t.*

Scaling. **Grid Hitting LP** is particularly useful since it allows to "convert" the cost of its objective function to a "relaxation" of its constraints. Roughly speaking, we delete vertices at a lower cost, but satisfy constraints that encode a larger treewidth. In particular, the tradeoff is quadratic. Formally, **Grid Hitting LP** has the scaling property stated in the following lemma.

Lemma 2.1. *Let α be a feasible fractional solution of* **Grid Hitting LP**(G, w, t) *for some triple (G, w, t), and let $s \in \mathbb{N}$ (where s can depend on (G, w, t)). Define α' by $\alpha'(x_v) = \alpha(x_v)/s^2$ for all $v \in V(G)$. Then, α' is a feasible fractional solution of* **Grid Hitting LP**$(G, w, s \cdot t)$ *such that $\texttt{cost}(\alpha') = \texttt{cost}(\alpha)/s^2$.*

Proof. To prove that α' is a feasible fractional solution of **Grid Hitting LP**(G, w, st), consider some set $S \in \texttt{Grid}_{st}(G)$. Then, there exists a partition $S = S_1 \uplus S_2 \uplus \cdots \uplus S_{s^2}$ such that $S_i \in \texttt{Grid}_t(G)$ for all $i \in \{1, 2, \ldots, s^2\}$. (To see this, let H be a grid, such that $V(H) = \{v_{i,j} : i, j \in \{1, 2, \ldots, st\}$ and $E(H) = \{\{v_{i,j}, v_{i',j'}\} : |i - i'| + |j - j'| = 1\}$. Consider a minor model φ of H in G. Then, for all $p, q \in \{1, 2, \ldots, s\}$, we have that φ restricted to $\{v_{i,j} : i \in \{(p-1)t+1, (p-1)t+2, \ldots, pt\}, j \in \{(q-1)t+1, (q-1)t+2, \ldots, qt\}\}$ is a minor model of a $t \times t$ grid in G.) Because α is a feasible fractional solution of **Grid Hitting LP**(G, w, t), we have that $\sum_{v \in S_i} \alpha(x_v) \geq 1$ for all $i \in \{1, 2, \ldots, s^2\}$. Therefore,

$$\sum_{v \in S} \alpha'(x_v) = \sum_{v \in S} \alpha(x_v)/s^2 = \sum_{i=1}^{s^2} (\sum_{v \in S_i} \alpha(x_v)/s^2) \geq \sum_{i=1}^{s^2} (1/s^2) = 1$$

Thus, α' is a feasible fractional solution of **Grid Hitting LP**(G, w, st). Now, note that

$$\mathtt{cost}(\alpha') = \sum_{v \in V(G)} w_v \alpha'(x_v) = \sum_{v \in V(G)} w_v \alpha(x_v)/s^2 = \mathtt{cost}(\alpha)/s^2.$$

This completes the proof. $\square$

2.3 Translation Between LPs

We first show how a feasible fractional solution of **Well-Linkedness LP** can be translated to a feasible fractional solution of **Grid Hitting LP**. In particular, we show the following.

Lemma 2.2. *There exists a fixed constant c such that for any triple (G, w, t) and feasible fractional solution α of* **Well-Linkedness LP**(G, w, t)*, the following claim holds. Define $\alpha' : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ by $\alpha'(x_v) = (1/t) \cdot \alpha(x_v)$ for all $v \in V(G)$. Then, α' is a feasible fractional solution of* **Grid Hitting LP**(G, w, ct) *such that* $\mathtt{cost}(\alpha') = (1/t) \cdot \mathtt{cost}(\alpha)$.

Observe that in Lemma 2.2, we "win" a factor of $1/t$ of the cost. Next, we need to go from a feasible fraction solution of **Grid Hitting LP**(G, w, t) to a feasible fraction solution of **Well-Linkedness LP**(G, w, t). This is achieved in two stages. We first go to what an LP that we call **Pairwise-Flow LP**.

(h, t)-Pairwise Flow and the Pairwise-Flow Hitting LP. For a graph G and an integer h, we let $\mathtt{ConPart}(G, h)$ denote the set of all tuples $(X_1, X_2, \ldots, X_h)$ of pairwise disjoint subsets of $V(G)$, where each X_i, $i \in \{1, 2, \ldots, h\}$, is a connected set in G. Having this notation, we define the following notion.

Definition 2.1. *A graph G has an (h, t)-*pairwise flow *if there exists $\overline{X} \in \mathtt{ConPart}(G, h)$ such that for all distinct $i, j \in \{1, 2, \ldots, h\}$, the maximum number of pairwise vertex-disjoint paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$ is lower bounded by t. The maximum integer t such that G has an (h, t)-pairwise flow is denoted by* $\mathtt{pf}_h(G)$.

In **Pairwise-Flow Hitting LP**, we are given a graph G that is H minor-free for a graph H with $|V(H)| = h$, a weight function $w : V(G) \to \mathbb{Q}^+$ and $t \in \mathbb{N}$. Our objective is to delete vertices of minimum weight so that G will not have an $(h, t+1)$-pairwise flow. Each variable x_v indicates whether we delete v. For every $\overline{X} \in \mathtt{ConPart}(G, h)$ and $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, we have variables $\lambda^{\overline{X},(i,j)}$ and $y_v^{\overline{X},(i,j)}$ for all $v \in V(G)$. Informally, setting the variable $\lambda^{\overline{X},(i,j)}$ to 1 indicates that the maximum number of pairwise vertex-disjoint paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$, after the deletion of vertices as indicated by the variables x_v for all $v \in V(G)$, should be upper bounded by t. In turn, the satisfaction of this indication is realized by using the variables $y_v^{\overline{X},(i,j)}$ for all $v \in V(G)$, which specify (by setting their value to 1) which vertices hit, together with the vertices already deleted, all paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$.

$$
\begin{array}{lll}
 & \textbf{Pairwise-Flow Hitting LP}(G,w,h,t)\textbf{:} & \\
\min & \displaystyle\sum_{v\in V(G)} w_v x_v & \\
\text{s.t.} & \displaystyle\sum_{i=1}^{h-1}\sum_{j=i+1}^{h} \lambda^{\overline{X},(i,j)} \geq 1 & \forall \overline{X} \in \mathtt{ConPart}(G,h) \\
 & \displaystyle\sum_{v\in V(G)} y_v^{\overline{X},(i,j)} \leq t & \forall \overline{X} \in \mathtt{ConPart}(G,h), i,j \in \{1,\ldots,h\}, i<j \\
 & \displaystyle\sum_{v\in V(P)} (x_v + y_v^{\overline{X},(i,j)}) \geq \lambda^{\overline{X},(i,j)} & \forall \overline{X} \in \mathtt{ConPart}(G,h), i,j \in \{1,\ldots,h\}, i<j, \\
 & & P \in \mathcal{P}_{G-(X\setminus(\overline{X}[i]\cup\overline{X}[j]))}(\overline{X}[i],\overline{X}[j]) \\
 & x_v \geq 0, y_v^{\overline{X},(i,j)} \geq 0, \lambda^{\overline{X},(i,j)} \geq 0 &
\end{array}
$$

The rationale behind the definition of this LP, in the context of our work, stems from the following result which we prove.

Lemma 2.3. *Let H be a graph and denote $h = |V(H)|$. Then, for any graph G that is H-minor free, it holds that $\mathtt{pf}_h(G) < 5\binom{h}{2}^3(\mathtt{tw}(G)+1)$.*

We are now ready to present the translation. Here, given a feasible fractional solution of **Grid Hitting LP**, the translation entails the multiplication of the value assigned to each variable x_v by $\mathcal{O}(t)$, along with the extension of the result to the variable set of **Pairwise-Flow Hitting LP**. We also pay the penalty of multiplying t by a fixed constant.

Lemma 2.4. *Let H be a graph with $h = |V(H)|$. There exist fixed constants $c = c(H)$ and $d = d(H)$ such that given any triple (G, w, t) where G is H-minor free, and given any feasible fractional solution α of* **Grid Hitting LP**(G, w, t)*, the following claim holds. Define α' by $\alpha'(x_v) = dt \cdot \alpha(x_v)$ for all $v \in V(G)$. Then, there exists a feasible fractional solution $\alpha^\star$ of* **Pairwise-Flow Hitting LP**(G, w, h, ct) *that extends α' and such that $\mathtt{cost}(\alpha^\star) = dt \cdot \mathtt{cost}(\alpha)$.*

Finally, we present the last translation. Here, given a feasible fractional solution of **Pairwise-Flow Hitting LP**, the translation entails the multiplication of the value assigned to each variable x_v by a fixed constant, along with the extension of the result to the variable set of **Well-Linkedness LP**. Again, we also pay the penalty of multiplying t by a fixed constant.

Lemma 2.5. *Let H be a graph with $h = |V(H)|$. There exist fixed constants $c = c(H)$ and d such that given any triple (G, w, t) where G is H-minor free, and given any feasible fractional solution α of* **Pairwise-Flow Hitting LP**(G, w, h, t)*, the following claim holds. Define α' by $\alpha'(x_v) = d \cdot \alpha(x_v)$ for all $v \in V(G)$. Then, there exists a feasible fractional solution $\alpha^\star$ of* **Well-Linkedness LP**(G, w, ct) *that extends α' and such that $\mathtt{cost}(\alpha^\star) = d \cdot \mathtt{cost}(\alpha)$.*

The proof of Lemma 2.5 is done into two phases. The objective of Phase I is to show that, for any infeasible fractional solution of **Well-Linkedness LP**, we can obtain a structured witness that is an adaptation of the following.

Definition 2.2. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. An (s, δ, λ)-*fractional well-linkedness witness *with respect to (G, x), or simply an (s, δ, λ)-*witness*, is a pair $(S, \mathcal{Q})$ where $S \subseteq V(G)$, $|S| = s$, and $\mathcal{Q}$ is a collection of paths in G that includes exactly one path in $\mathcal{P}(u, v)$ for each pair of vertices $u, v \in S$, such that the following conditions hold.*

1. *For every path $P \in \mathcal{Q}$, it holds that $\sum_{v \in V(P)} x_v \leq \delta$.*
2. *For every vertex $v \in V(G)$, it holds that $\lambda \cdot |\{P \in \mathcal{Q} : v \in V(P)\}| \leq 1$.*

The existence of this witness will be the foundation of Phase II, where the actual translation is made. We begin by modifying Definition 2.2 in two ways: first, we need to allow having, between every pair of vertices, many fractional paths rather than only one fractional path; second, we do not obtain a clique but only ensure that between any two large subsets of our structure, the flow is large. From this, we proceed to the actual translation in Phase II.

2.4 Rounding the Well-Linkedness LP

Finally, we give our rounding algorithm. This shows that on H-minor free graphs the integrality gap for **Well-Linkedness LP** is $\mathcal{O}(\log n)$. In particular, we show the following theorem.

Theorem 2.1. *There is a polynomial time algorithm that, for any graph H, takes as input an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$, and an integer t, and outputs a vertex set S such that $\mathtt{tw}(G) \leq \mathcal{O}(t)$ and $w(S)$ is at most $\mathcal{O}(\log n)$ times the optimum of* **Well-Linkedness LP**(G, w, t)*. That is, $w(S) \leq \mathcal{O}(\log n \cdot \mathtt{opt}(G, w, t))$.*

The proofs of Lemmas 2.2, 2.4 and 2.5 and Theorem 2.1 require several novel ideas and are quite technical. We invite the readers to read the corresponding subsections for more details. For example, for the proof of Theorem 2.1 we need to use the well-known Klein-Plotkin-Rao rounding scheme as well as a new "weight redistribution" trick.

2.5 Summary

Recall that we have two LPs: **Well-Linkedness LP** and **Grid Hitting LP**. One has a good LP rounding property, and the other has a good scaling property. Now we need to translate solutions of one LP to other and vice-versa. The polynomial time algorithm described in Theorem 1.1 has the following steps. Given an input (G, w, t) it works as described below. Let c_1, c_2, d_1, d_2 be some fixed constants that only depend on H.

Step A. Solve **Well-Linkedness LP** on (G, w, t) and obtain a solution α. We must remark here that we cannot solve this LP, and we need to apply the methodology of *round and separate* introduced by Bansal et al. [7].

Step B. Translate α into a fractional solution α_1 of **Grid Hitting LP**$(G, w, c_1 t)$ such that $\mathtt{cost}(\alpha_1) = (1/t) \cdot \mathtt{cost}(\alpha)$. (Lemma 2.2.)

Step C. Then, scale the feasible solution α_1 of **Grid Hitting LP**$(G, w, c_1 t)$ to get a feasible fractional solution α_2 of **Grid Hitting LP**$(G, w, s \cdot c_1 t)$ such that $\mathtt{cost}(\alpha_2) = \mathtt{cost}(\alpha_1)/s^2$. (Lemma 2.1.)

Step D. Now given a feasible fractional solution α_2 of **Grid Hitting LP**$(G, w, s \cdot c_1 t)$ such that $\mathtt{cost}(\alpha_2) = \mathtt{cost}(\alpha_1)/s^2$, we obtain a feasible fractional solution $\alpha^\star$ of **Well-Linkedness LP** on $(G, w, s \cdot c_2 c_1 t)$ of cost $d_1 st \cdot \mathtt{cost}(\alpha_2)$. (Lemmas 2.4 and 2.5.)

Step E. Finally, we round $\alpha^\star$ (using Theorem 2.1) to get an integral solution $\alpha^\dagger$ that corresponds a subset $S \subseteq V(G)$ such that $\mathtt{tw}(G - S) \leq c \cdot st$, and

$$\begin{aligned}
\texttt{cost}(\alpha^\dagger) &\leq d_2 \log n \cdot \texttt{cost}(\alpha^\star) \\
&\leq d_2 \log n \cdot d_1 st \cdot \texttt{cost}(\alpha_2) \\
&\leq d_2 \log n \cdot d_1 st \cdot \frac{\texttt{cost}(\alpha_1)}{s^2} \\
&\leq d_2 \log n \cdot d_1 st \cdot \frac{\texttt{cost}(\alpha)}{s^2 t} \\
&= d_2 \log n \cdot d_1 \cdot \frac{\texttt{cost}(\alpha)}{s} \\
&\leq d_2 \log n \cdot d_1 \cdot \frac{\texttt{opt}(G, w, t)}{s}.
\end{aligned}$$

Thus, these steps together yield a proof of Lemma 1.1.

3 Preliminaries

Given a set X, we use $X = X_1 \uplus X_2 \uplus \cdots \uplus X_h$ to denote that $\overline{X} = (X_1, X_2, \ldots, X_h)$ is a partition of X. Moreover, we use $\overline{X}[i]$ to denote X_i for all $i \in \{1, 2, \ldots, h\}$. Let $\mathbb{Q}_0^+$ contain all non-negative rational numbers.

Graphs. Wherever it is not explicitly written otherwise, we consider undirected graphs. Given a graph G, we let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Given a collection of graph $\mathcal{G}$, we denote $V(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} V(G)$. Given a vertex $v \in V(G)$, denote the open and closed neighborhoods of v in G by $N_G(v)$ and $N_G[v]$, respectively. Moreover, given a subset $U \subseteq V(G)$, denote $N_G[U] = \bigcup_{v \in U} N_G[v]$ and $N_G(U) = N_G[U] \setminus U$. Given $A, B \subseteq V(G)$, denote $E_G(A, B) = \{\{a, b\} \in E(G) : a \in A, b \in B\}$. Given a function $f : V(G) \to \mathbb{Q}$, we denote $f_v = f(v)$. Given two vertices $u, v \in V(G)$, we let $\mathcal{P}_G(u, v)$ denote the collection of (simple) paths in G whose endpoints are u and v, and let $\mathcal{W}_G(u, v)$ denote the collection of walks in G whose endpoints are u and v. When G is clear from context, we drop it from the subscript. Given two paths P and P' that have one common endpoints, by PP' we denote that walk that results from the concatenation of P and P'. Moreover, given subsets $A, B \subseteq V(G)$, denote $\mathcal{P}_G(A, B) = \bigcup_{u \in A, v \in B} \mathcal{P}_G(u, v)$. Given a subset $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of G induced by U, and we denote $G - U = G[V(G) \setminus U]$. We say that U is a *connected set* if $G[U]$ is a connected graph. Given a subset $U \subseteq E(G)$, we let $G - U$ denote the graph on $V(G)$ with the edge set $E(G) \setminus U$, and $G[U] := (V(G), U)$. An *α-separator* of a subset $S \subseteq V(G)$ in G is a subset $X \subseteq V(G)$ such that for every connected component C of $G - X$ it holds that $|V(C) \cap S| \leq \alpha |S|$. A *$t \times t$-grid* is a graph H whose vertex set can be denoted by $\{v_{i,j} : i, j \in \{1, 2, \ldots, t\}\}$ so that $E(H) = \{\{v_{i,j}, v_{i',j'}\} : |i - i'| + |j - j'| = 1\}$.

For a directed graph D, we let $A(D)$ denote its arc set. Given a vertex $v \in V(D)$, we let $N_{\text{in}}(v) := \{u \in V(D) : (u, v) \in A(D)\}$ denote the ingoing neighborhood of v, and $N_{\text{out}}(v) := \{u \in V(D) : (v, u) \in A(D)\}$ denote the outgoing neighborhood of v.

Given a graph G and an edge $e = \{u, v\} \in E(G)$, the graph obtained by contracting e in G is the graph whose vertex set is $(V(G) \setminus \{u, v\}) \cup \{x\}$ for some new vertex x, with the edges in $\{\{a, b\} \in E(G) : a, b \notin \{u, v\}\}$, and with a new edge $\{x, y\}$ for every edge $\{y, v\} \in E(G)$ and $\{y, u\} \in E(G)$. Note that this operation may result in multiedges. Throughout the paper, we implicitly assume that multiplicities of edges are automatically reduced to 1, thus we deal with simple graphs, unless explicitly stated otherwise (which will be the case when we discuss duals of bounded genus graphs later). We let G/e denote the result of contracting e in G. Given a connected set $U \subseteq V(G)$, we let G/U denote the graph obtained by contracting the edges of some spanning tree of $G[U]$—specifically, the resulting graph has the vertex set $(V(G)\setminus U)\cup\{x\})$ for some new vertex x and with a new edge $\{x,y\}$ for every edge $\{y,v\}\in E(G)$ such that $v\in U$. Given a subset $U\subseteq V(G)$, we let G/U denote the result of the contraction of every maximal connected subset of U. Note that the contraction of each individual connected subset $T\subseteq U$ results one vertex in $v\in V(G/U)\setminus V(G)$; we refer to the set T the *origin of* v, and denote $\mathsf{Origin}_{G,U}(v)=T$. For vertices $v\in V(G/U)\cap V(G)$, denote $\mathsf{Origin}_{G,U}(v)=\{v\}$. When, G and U are clear from context, we drop the subscript.

Minors, treewidth, and well-linkedness. We say that a graph H is a *minor* of a graph G if there exists a function $\varphi: V(H)\to 2^{V(G)}$ such that for all $v\in V(H)$, it holds that $G[\varphi(v)]$ is connected, for all $u,v\in V(H)$, it holds that $\varphi(u)\cap\varphi(v)=\emptyset$, and for all $\{u,v\}\in E(H)$, it holds that there exist $u'\in\varphi(u)$ and $v'\in\varphi(v)$ such that $\{u',v'\}\in E(G)$. Such a function φ is a *minor model* of H in G. Equivalently, a graph H is a minor of a graph G if H can be obtained from G by deleting vertices, deleting edges, and contracting edges. Given an integer $t\in\mathbb{N}$, we let $\mathtt{Grid}_t(G)$ denote the collection of all subsets $U\subseteq V(G)$ such that $G[U]$ contains a $t\times t$-grid as a minor.

Treewidth is a measure of how "treelike" is a graph, which is formally defined as follows.

Definition 3.1. *A* tree decomposition *of a graph G is a pair (T,β) of a tree T and $\beta: V(T)\to 2^{V(G)}$, such that*

1. *for any edge $\{x,y\}\in E(G)$ there exists a node $v\in V(T)$ such that $x,y\in\beta(v)$, and*
2. *for any vertex $x\in V(G)$, the subgraph of T induced by the set $T_x=\{v\in V(T): x\in\beta(v)\}$ is a non-empty tree.*

The width *of (T,β) is $\max_{v\in V(T)}\{|\beta(v)|\}-1$. The* treewidth *of G is the minimum width over all tree decompositions of G.*

Given a tree decomposition (T,β) of a graph G, for every $v\in V(T)$, the set $\beta(v)$ is called the *bag* of v, and we let $\gamma(v)$ denote the union of all the bags of the descendants of v in T including the bag of v. For the root r of T, the set $\beta(r)$ is called the *root bag*.

Note that if a graph H is a minor of a graph G, then $\mathtt{tw}(H)\leq\mathtt{tw}(G)$. Tightly linked to treewidth are grid minors, a relation that is discussed later. In addition, tightly linked to treewidth is also the notion of well-linkedness, which is defined as follows.

Definition 3.2. *Let G be a graph. A vertex set $S\subseteq V(G)$ is t-*linked *in G if S does not have a $\frac{1}{2}$-separator X in G with $|X|<t$. Moreover, the* linkedness *of G, denoted by $\mathtt{link}(G)$, is the maximum integer t such that there exists a t-linked set in G.*

H-Minor freeness, genus, and planarity. For a graph H, we say that a graph G is *H-minor free* if it does not contain H as a minor. A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. By Wagner's theorem, a graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as minors. A *plane graph* is a planar graph with a fixed embedding.

An embedding is called *cellular* if every face is homeomorphic to an open disk. The *genus* of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. A graph of *genus g* has a (cellular) embedding on a surface of genus g. In this paper, we consider only such cellular embeddings. A graph of genus 0 is a planar graph. Graphs of bounded genus g can be characterized as the graphs excluding a finite set (whose size depends on g) of graphs as minors, and are therefore H-minor free for any choice of H from that set.

Let G be a graph (cellularly) embedded on a surface. The *dual graph* D of G is the embedded graph whose vertex set includes a vertex v_f for each face f of G, and which contains an edge between v_f and $v_{f'}$ if the faces f and f' are adjacent (in G). Note that even if G is simple, its dual D may contain multiedges. Moreover, if G is a graph of genus g, then its dual D is also a graph of genus g, and the dual graph of D is G itself. For any edge $\{u,v\} \in E(G)$, we let $\varphi(\{u,v\})$ denote the edge $\{v_f, v_{f'}\}$ of the dual graph D of G such that f and f' are the (unique) faces in G having $\{u,v\}$ as a common boundary.

Let us explicitly state the following simple observation.

Observation 3.1 (Folklore)**.** *Let G be a graph (cellularly) embedded on a surface, and let D be the dual graph of G. Let $e \in E(G)$. Then, the dual of G/e is $G - \{\varphi(e)\}$.*

We also need the following proposition, relating the treewidth of G to the treewidth of D.

Proposition 3.1 ([55])**.** *There exists a fixed constant c such that for any graph G of genus g cellularly embedded on a surface, the treewidth of the dual graph D of G is upper bounded by $c \cdot (\mathtt{tw}(G) + g)$.*

Linear Programming (LP). An LP consists of a set of variables, say $X = \{x_1, x_2, \ldots, x_n\}$, an objective function $\sum_{i=1}^{n} c_i x_i$ for some constants c_i where the objective can be either maximization or minimization, and a set of m constraints: For $j = 1, 2, \ldots, m$, we have a constraint of the form $\sum_{i=1}^{n} a_{i,j} x_i \ \mathsf{sign}\ b_j$ for some constants $a_{i,j}$ and b_j and where $\mathsf{sign} \in \{\leq, =, \geq\}$.

A *fractional solution* of an LP is a function α that assigns a rational number to each variable. We say that a fractional solution is *feasible* if all of the constraints are satisfied; otherwise (that is, if at least one constraint is not satisfied), it is *infeasible*. We say that α is *integral* if its image is a subset of $\mathbb{Z}$. The cost of a fractional solution α, denoted by $\mathtt{cost}(\alpha)$, is defined as $\sum_{i=1}^{n} c_i \alpha(x_i)$. The optimum of the LP is the maximum or minimum cost—depending on whether its objective is maximization or minimization—of a feasible fractional solution of it (if one exists). An optimal (feasible fractional) solution is a feasible fractional solution whose cost is the optimum.

A *separating hyperplane* for an infeasible solution α to an LP is a function β that assigns a rational number to every variable, and a rational c, such that $\sum_{i=1}^{n} \alpha(x_i)\beta(x_i) > c$, and for every feasible solution α' we have $\sum_{i=1}^{n} \alpha'(x_i)\beta(x_i) \leq c$.

Given a maximization LP $\mathbf{P}$ with an objective function $\sum_{i=1}^{n} c_i x_i$, for $j = 1, 2, \ldots, m$, a constraint $\sum_{i=1}^{n} a_{i,j} x_i \leq b_j$, and for $i = 1, 2, \ldots, n$, the constraint $x_i \geq 0$, the *dual LP* of $\mathbf{P}$ is the minimization LP $\mathbf{D}$ defined as follows. The objective function of $\mathbf{D}$ is $\sum_{j=1}^{m} b_j y_j$, for $i = 1, 2, \ldots, n$, it has the constraint $\sum_{j=1}^{m} a_{i,j} y_j \geq c_i$, and for $j = 1, 2, \ldots, m$, it has the constraint $y_j \geq 0$. The dual of a minimization LP $\mathbf{D}$ is defined analogously. In particular, the dual of the dual of an LP $\mathbf{Q}$ is $\mathbf{Q}$ itself.

We now state the strong duality theorem.

Proposition 3.2 ([49])**.** *For any LP $\mathbf{Q}$, the optimum of $\mathbf{Q}$ and the optimum of the dual of $\mathbf{Q}$ are equal.*

Boundaried Graphs and Their Folios and Contractions. Roughly speaking, a boundaried graph is a graph where some vertices are labeled. Formally,

Definition 3.3. *A* boundaried graph *is a graph G with a set $\delta(G) \subseteq V(G)$ of distinguished vertices called* boundary vertices*, and a labeling $\lambda_G : \delta(G) \to 2^{\mathbb{N}} \setminus \{\emptyset\}$ such that for all distinct $u, v \in \delta(G)$, it holds that $\lambda_G(u) \cap \lambda_G(v) = \emptyset$. The set $\delta(G)$ is the* boundary *of G, and the* label set *of G is $\Lambda(G) = \{\lambda_G(v) : v \in \delta(G)\}$.*

Note that any (unboundaried) graph can be viewed as a boundaried graph with an empty boundary. For a boundaried graph G, we will only contract sets of vertices that contain no vertex in $\delta(G)$; then, $\delta(G/S) = \delta(G)$ and $\lambda_{G/S} = \lambda_G$.

We say that a boundaried graph H is a minor of a boundaried graph G if $\bigcup \Lambda(H) \subseteq \bigcup \Lambda(G)$ and there exists a function $\varphi : V(H) \to 2^{V(G)}$ such that *(i)* for all $v \in V(H)$, it holds that $G[\varphi(v)]$ is connected, *(ii)* for all $u, v \in V(H)$, it holds that $\varphi(u) \cap \varphi(v) = \emptyset$, *(iii)* for all $\{u, v\} \in E(H)$, it holds that there exist $u' \in \varphi(u)$ and $v' \in \varphi(v)$ such that $\{u', v'\} \in E(G)$, *(iv)* for all $v \in \delta(H)$, it holds that $\lambda_H(v) = \bigcup\{\lambda_G(v') : v' \in \varphi(v) \cap \delta(G)\}$, and *(v)* for all $v \in V(H) \setminus \delta(H)$, it holds that $\varphi(v) \cap \delta(G) = \emptyset$. Such a function φ is a *minor model* of H in G. Accordingly, the *ρ-folio* of a boundaried graph G is the set of all boundaried graphs on at most ρ vertices that are minors of G.

Given a boundaried graph G and a subset $U \subseteq V(G)$, the boundaried graph G/U is the graph G/U whose boundary consists of of every vertex $v \in \delta(G) \cap V(G/U)$ (whose label is $\lambda_{G/U}(v) = \lambda_G(v)$) and every vertex $v \in V(G/U) \setminus V(G)$ that originated from the contraction of a connected set X with non-empty intersection with $\delta(G)$ (then, $\lambda_{G/U}(v) = \bigcup_{u \in X \cap \delta(G)} \lambda_G(u)$).

4 From Well-Linkedness LP to Grid Hitting LP

In this section, we translate a feasible fractional solution of an LP called **Well-Linkedness LP** (by Bansal et al. [7]) to a feasible fractional solution of an LP called **Grid Hitting LP**. In addition, we exhibit the scaling property of the **Grid Hitting LP**.

4.1 Well-Linkedness LP

The (edge version of) **Well-Linkedness LP** is due to Bansal et al. [7]. Here, we present the vertex version of this LP. However, their proof of the dual of an LP nested inside **Well-Linkedness LP** (as explained later) directly extends to the vertex version.[3]

In this **Well-Linkedness LP**, we are given a graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and $t \in \mathbb{N}$. Our objective is to delete vertices of minimum weight so that G will not have a $(t+1)$-linked set. Each variable x_v indicates whether we delete v. For every subset $S \subseteq V(G)$, we have a variable y^S_v for all $v \in V(G)$, and a variable d^S_{uv} for all $u, v \in S$. Informally, these variables exhibit a $\frac{1}{2}$-separator Y of S in G with $|Y| \leq t$ as follows. First, each variable y^S_v indicates whether v belongs to Y. Now, each variable d^S_{uv} encodes the distance (lightest path) between u and v in the metric where the weight of each vertex r is given by $(x_r + y^S_r)$. In the constraints, we have that $d^S_{uv} \leq \sum_{r \in V(P)}(x_r + y^S_r)$ for all $P \in \mathcal{P}_G(u, v)$, and since it is "beneficial" too keep d^S_{uv} as large as possible, this means that it can be thought of as being equal to the distance $\sum_{r \in V(P)}(x_r + y^S_r)$ above. Now, to ensure that Y is a $\frac{1}{2}$-separator of S in G, it should (roughly) hold that in this metric, for every vertex $u \in S$ that can exist at most $\frac{|S|}{2}$ vertices at distance smaller than 1 from u. This is encoded by having a constraint $\sum_{v \in U} d^S_{uv} \geq |U| - \frac{|S|}{2}$ for all $U \subseteq S \subseteq V(G)$ and $u \in U$. Here, the consideration of every subset $U \subseteq S$ aims to replace the need to cap (bound from above) the distances d^S_{uv} by 1 and hence lose the property of encoding a metric. Specifically, for a vertex $u \in S$, our purpose is to satisfy the constraint where U is the set of all vertices at distance at most 1 from u.

[3] We note that, however, their bicriteria $(\log n \log\log n, \log \eta)$-approximation algorithm (on general graphs) does not extend to the edge version of Bounded Treewidth Interdiction (without using our "weight redistribution" trick).

$$
\begin{array}{lll}
 & \textbf{Well-Linkedness LP}(G,w,t)\textbf{:} & \\
\min & \sum_{v\in V(G)} w_v x_v & \\
\text{s.t.} & \sum_{v\in V(G)} y_v^S \leq t & \forall S \subseteq V(G) \\
 & d_{uv}^S \leq \sum_{r\in V(P)} (x_r + y_r^S) & \forall S \subseteq V(G), u,v \in S, P \in \mathcal{P}_G(u,v) \\
 & \sum_{v\in U} d_{uv}^S \geq |U| - \frac{|S|}{2} & \forall U \subseteq S \subseteq V(G), u \in U \\
 & x_v \geq 0, y_v^S \geq 0, d_{uv}^S \geq 0 &
\end{array}
$$

The rationale behind the definition of this LP, in the context of our work, stems from the following known results. First, it is tightly linked to treewidth:

Proposition 4.1 ([52]). *For any graph G, it holds that $\mathtt{link}(G) < \mathtt{tw}(G) \leq 4\mathtt{link}(G)$.*

In addition, Bansal et al. [7] showed how this LP can be "partially solved", as explained in the following subsection.

4.2 Duality of Well-Linkedness LP

Let us begin by presenting the (known) dual LP of an LP that is "nested" inside **Well-Linkedness LP**. We will need to analyze this dual LP to prove both our transition from **Well-Linkedness LP** to Grid Hitting LP and our transition from **Pairwise-Flow Hitting LP** to **Well-Linkedness LP**. (In each transition, the analysis and properties exposed are different). First, let us explicitly state this "nested" LP. To be consistent with Bansal et al. [7], we refer to this LP as **sep-LP**. In this LP, we are given a graph G, a subset $S \subsetneq V(G)$ and a function $x : V(G) \to \mathbb{Q}_0^+$. We stress that x_v, for any $v \in V(G)$, is not a variable. Here, we have a variable y_v for all $v \in V(G)$, and a variable d_{uv}^S for all $u, v \in S$. Note that these variables have the exact same meaning as in **Well-Linkedness LP**; specifically, they exhibit a $\frac{1}{2}$-separator Y of S in G.

$$
\begin{array}{lll}
 & \textbf{sep-LP}(G,S,x)\textbf{:} & \\
\min & \sum_{v\in V(G)} y_v & \\
\text{s.t.} & d_{uv} \leq \sum_{r\in V(P)} (x_r + y_r) & \forall u,v \in S, P \in \mathcal{P}_G(u,v) \\
 & \sum_{v\in U} d_{uv} \geq |U| - \frac{|S|}{2} & \forall U \subseteq S, u \in U \\
 & y_v \geq 0, d_{uv} \geq 0 &
\end{array}
$$

By **sep-LP**(G, S, α) where $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$, we refer to **sep-LP**$(G, S, x')$ where $x'_v = \alpha(x_v)$ for every $v \in V(G)$. The following observation is immediate given the formulations of **Well-Linkedness LP** and **sep-LP**.

Observation 4.1. *Let G be a graph, $w : V(G) \to \mathbb{Q}^+$, $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ and $t \in \mathbb{N}$. Then, α can be extended to a feasible fractional solution of* **Well-Linkedness LP**(G, w, t) *if* *and only if for every subset $S \subseteq V(G)$, α can be extended to a feasible fraction solution of* **sep-LP**(G, S, α) *of cost at most t.*

By Bansal et al. [7] the dual of **sep-LP**, called **flow-LP**, is the LP given below. In this LP, we are given a graph G, a subset $S \subseteq V(G)$ and a function $x : V(G) \to \mathbb{Q}_0^+$. Again, we stress that x_v, for any $v \in V(G)$, is not a variable. We remark that Bansal et al. [7] examined this LP to present a result regarding the partial solution of **Well-Linkedness LP** (required for rounding). We need this LP also for very different purposes (and therefore make very different use of it in the analysis of our transitions). For our purposes, we will need to give our (rough) intuitive interpretation of it when it is required (in Section 6.1). For now, we simply state it, and remark that the variable set includes a variable $g_{U,v}$ for all $U \subseteq S$ and $v \in U$, and a variable f_P^{uv} for all $u, v \in S$ and $P \in \mathcal{P}(u, v)$.

flow-LP(G, S, x)**:**

$$
\begin{aligned}
\max \quad & \sum_{U \subseteq S} \sum_{v \in U} g_{U,v}(|U| - \frac{|S|}{2}) - \sum_{u,v \in S} \sum_{P \in \mathcal{P}(u,v)} f_P^{uv} \left(\sum_{r \in V(P)} x_r \right) \\
\text{s.t.} \quad & \sum_{\substack{U \subseteq S \\ \text{s.t. } u,v \in U}} (g_{U,u} + g_{U,v}) \leq \sum_{P \in \mathcal{P}(u,v)} f_P^{uv} & \forall u, v \in S \\
& \sum_{u,v \in S} \sum_{\substack{P \in \mathcal{P}(u,v) \\ \text{s.t. } r \in V(P)}} f_P^{uv} \leq 1 & \forall r \in V(G) \\
& g_{U,v} \geq 0, f_P^{uv} \geq 0
\end{aligned}
$$

From Proposition 3.2, we obtain the following observation.

Observation 4.2. *Let G be a graph, $S \subseteq V(G)$ and $x : V(G) \to \mathbb{Q}_0^+$. Then, the optimum of* **sep-LP**(G, S, x) *is equal to the optimum of* **flow-LP**(G, S, x).

The following exposition towards the statement of Proposition 4.2 will be relevant to Section 7.[4] While **sep-LP**(G, S, x) has an exponential number of constraints, Bansal et al. [7] observed that it can be solved in polynomial time. Roughly speaking, the idea is that explicit tests for the satisfaction of the constraint for every individual path $P \in \mathcal{P}_G(u, v)$ can be replaced by a computation of distance (e.g., by calling Dijkstra's algorithm), and that explicit tests for the satisfaction of the constraints for every individual subset $U \subseteq S$ are not required—it suffices to make a test only for the subset $U \subseteq S$ that includes all vertices at distance at most 1 from the reference vertex u. Given a graph G, $w : V(G) \to \mathbb{Q}^+$, $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$, $t \in \mathbb{N}$ and a subset $S \subseteq V(G)$, if α cannot be extended to a feasible fraction solution of **sep-LP**(G, S, α) of cost at most t (which can be tested in polynomial time by the discussion above), then Observation 4.1 implies that α is infeasible for **Well-Linkedness LP**(G, w, t). In this case, Bansal et al. [7] rely on the dual LP, **flow-LP**(G, S, x), to find a separating hyperplane that witnesses that all extensions of α are infeasible for **Well-Linkedness LP**(G, w, t). We summarize this discussion with the following proposition, whose proof can be found in [7].

Proposition 4.2 ([7])**.** *There is a polynomial time algorithm that given a graph G, $w : V(G) \to \mathbb{Q}^+$, $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$, $t \in \mathbb{N}$ and a subset $S \subseteq V(G)$, either finds* (i) *a separating hyperplane witnessing that every extension of α is infeasible for* **Well-Linkedness LP**(G, w, t)*,*[5]

[4] Nevertheless, we give the discussion in this subsection since it concerns **flow-LP**(G, S, x), while in Section 7 we will not need to explicitly refer to **flow-LP**(G, S, x) at all.

[5] In this case, as shown by Bansal et al. [7], the algorithm is guaranteed to find a separating hyperplane that assigns 0 to every variable of **Well-Linkedness LP**(G, w, t) that is not in $\{x_v : v \in V(G)\}$.

or (ii) *a feasible solution* $\beta^S : \{y_v : v \in V(G)\} \cup \{d_{uv} : u, v \in V(G)\} \to \mathbb{Q}_0^+$ *to* **sep-LP**(G, S, α) *of cost at most* t*.*

4.3 Grid Hitting LP and its scaling property

In **Grid Hitting LP**, we are given a graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and $t \in \mathbb{N}$. Our objective is to hit all $t \times t$-grids in G. Each variable x_v indicates whether we delete v, and the constraints are self-explanatory.

Grid Hitting LP(G, w, t)**:**

$$\begin{aligned} \min \quad & \sum_{v \in V(G)} w_v x_v & \\ \text{s.t.} \quad & \sum_{v \in S} x_v \geq 1 & \forall S \in \mathtt{Grid}_t(G) \\ & x_v \geq 0 & \end{aligned}$$

The rationale behind the definition of this LP, in the context of our work, stems from the following known results. On the one hand, we have the following relation.

Proposition 4.3 ([23])**.** *Let H be a graph. There exists a fixed constant $c = c(H)$ such that for any H-minor free graph G of treewidth lower bounded by ct, it holds that G has a $t \times t$-grid as a minor.*

On the other hand, the following result implies that if G has a $t \times t$-grid as a minor, then its treewidth is lower bounded by t.

Proposition 4.4 ([15])**.** *The treewidth of a $t \times t$ grid is exactly t.*

Scaling. **Grid Hitting LP** is particularly useful since it allows to "convert" the cost of its objective function to a "relaxation" of its constraints. Roughly speaking, we delete vertices at a lower cost, but satisfy constraints that encode a larger treewidth. In particular, the tradeoff is quadratic. Formally, **Grid Hitting LP** has the scaling property stated in the following lemma.

Lemma 4.1. *Let α be a feasible fractional solution of* **Grid Hitting LP**(G, w, t) *for some triple (G, w, t), and let $s \in \mathbb{N}$ (where s can depend on (G, w, t)). Define α' by $\alpha'(x_v) = \alpha(x_v)/s^2$ for all $v \in V(G)$. Then, α' is a feasible fractional solution of* **Grid Hitting LP**$(G, w, s \cdot t)$ *such that* $\mathtt{cost}(\alpha') = \mathtt{cost}(\alpha)/s^2$.

Proof. To prove that α' is a feasible fractional solution of **Grid Hitting LP**(G, w, st), consider some set $S \in \mathtt{Grid}_{st}(G)$. Then, there exists a partition $S = S_1 \uplus S_2 \uplus \cdots \uplus S_{s^2}$ such that $S_i \in \mathtt{Grid}_t(G)$ for all $i \in \{1, 2, \ldots, s^2\}$. (To see this, let H be a grid, such that $V(H) = \{v_{i,j} : i, j \in \{1, 2, \ldots, st\}$ and $E(H) = \{\{v_{i,j}, v_{i',j'}\} : |i - i'| + |j - j'| = 1\}$. Consider a minor model φ of H in G. Then, for all $p, q \in \{1, 2, \ldots, s\}$, we have that φ restricted to $\{v_{i,j} : i \in \{(p-1)t+1, (p-1)t+2, \ldots, pt\}, j \in \{(q-1)t+1, (q-1)t+2, \ldots, qt\}\}$ is a minor model of a $t \times t$ grid in G.) Because α is a feasible fractional solution of **Grid Hitting LP**(G, w, t), we have that $\sum_{v \in S_i} \alpha(x_v) \geq 1$ for all $i \in \{1, 2, \ldots, s^2\}$. Therefore,

$$\sum_{v \in S} \alpha'(x_v) = \sum_{v \in S} \alpha(x_v)/s^2 = \sum_{i=1}^{s^2} (\sum_{v \in S_i} \alpha(x_v)/s^2) \geq \sum_{i=1}^{s^2} (1/s^2) = 1$$

Thus, α' is a feasible fractional solution of **Grid Hitting LP**(G, w, st). Now, note that

$$\texttt{cost}(\alpha') = \sum_{v \in V(G)} w_v \alpha'(x_v) = \sum_{v \in V(G)} w_v \alpha(x_v)/s^2 = \texttt{cost}(\alpha)/s^2.$$

This completes the proof. $\square$

4.4 Translation

Towards the translation of a feasible fractional solution of **Well-Linkedness LP** to a feasible fractional solution of **Grid Hitting LP**, we need to establish several claims. For this purpose, let us first introduce a structure that we call a *fractional well-linkedness witness.*

Definition 4.1. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. An (s, δ, λ)-*fractional well-linkedness witness *with respect to (G, x), or simply an (s, δ, λ)-*witness*, is a pair $(S, \mathcal{Q})$ where $S \subseteq V(G)$, $|S| = s$, and $\mathcal{Q}$ is a collection of paths in G that includes exactly one path in $\mathcal{P}(u, v)$ for each pair of vertices $u, v \in S$, such that the following conditions hold.*

1. *For every path $P \in \mathcal{Q}$, it holds that $\sum_{v \in V(P)} x_v \leq \delta$.*
2. *For every vertex $v \in V(G)$, it holds that $\lambda \cdot |\{P \in \mathcal{Q} : v \in V(P)\}| \leq 1$.*

The term (s, δ, λ)-witness with respect to (G, α), where $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$, is an abbreviation of an (s, δ, λ)-witness with respect to (G, x') where $x'_v = \alpha(x_v)$ for every $v \in V(G)$. Now, we show that the existence of a certain fractional well-linkedness witness guarantees that the value of the objective function of **flow-LP**(G, S, x) is large. Later, in Section 6, we also show that if the value of **flow-LP**(G, S, x) is large, then there exists a certain fractional well-linkedness witness.

Lemma 4.2. *Let G be a graph, $S \subseteq V(G)$ and $x : V(G) \to \mathbb{Q}_0^+$. Let $\mathcal{Q}$ and δ be such that $(S, \mathcal{Q})$ is an $(s, \delta, 1/(2s))$-witness. Then, there exists a feasible fractional solution α of* **flow-LP**(G, S, x) *with* $\texttt{cost}(\alpha) \geq (\frac{1}{8} - \frac{\delta}{2})s$.

Proof. For any pair of vertices $u, v \in S$, let Q_{uv} be the unique path in $\mathcal{Q}$ between them. Define a function $\alpha : \{g_{U,v} : U \subseteq S, v \in U\} \cup \{f_P^{uv} : u, v \in S, P \in \mathcal{P}(u, v)\} \to \mathbb{Q}_0^+$ as follows.

- For all $U \subset S$ $(U \neq S)$ and $v \in U$, define $\alpha(g_{U,v}) = 0$.
- For all $v \in S$, define $\alpha(g_{S,v}) = 1/(4s)$.
- For all $u, v \in S$ and $P \in \mathcal{P}(u, v) \setminus \mathcal{Q}$, define $\alpha(f_P^{uv}) = 0$.
- For all $u, v \in S$ and $P \in \mathcal{P}(u, v) \cap \mathcal{Q}$ (note that P is unique), define $\alpha(f_P^{uv}) = 1/(2s)$.

Let us verify that α is a feasible fractional solution of **flow-LP**(G, S, x). We first show that $\sum_{\substack{U \subseteq S \\ \text{s.t. } u,v \in U}} (\alpha(g_{U,u}) + \alpha(g_{U,v})) \leq \sum_{P \in \mathcal{P}(u,v)} \alpha(f_P^{uv})$ for all $u, v \in S$. To this end, consider some vertices $u, v \in S$. Then, we have that

$$\begin{aligned} \sum_{\substack{U \subseteq S \\ \text{s.t. } u,v \in U}} (\alpha(g_{U,u}) + \alpha(g_{U,v})) &= \alpha(g_{S,u}) + \alpha(g_{S,v}) = 1/(4s) + 1/(4s) \\ &= 1/(2s) = \alpha(f_{Q_{uv}}^{uv}) = \sum_{P \in \mathcal{P}(u,v)} \alpha(f_P^{uv}). \end{aligned}$$

Second, we show that $\sum_{u,v\in S}\sum_{\substack{P\in\mathcal{P}(u,v)\\ \text{s.t. } r\in V(P)}} \alpha(f_P^{uv}) \leq 1$ for all $r \in V(G)$. To this end, consider some vertex $r \in V(G)$. Since $(S, \mathcal{Q})$ is an $(s, \delta, 1/(2s))$-witness, we have that

$$\sum_{u,v\in S}\sum_{\substack{P\in\mathcal{P}(u,v)\\ \text{s.t. } r\in V(P)}} \alpha(f_P^{uv}) = 1/(2s)\cdot|\{P\in\mathcal{Q} : r\in V(P)\}| \leq 1.$$

Thus, α is indeed a feasible fractional solution of **flow-LP**(G, S, x). Let us now analyze its cost. Since $(S, \mathcal{Q})$ is an $(s, \delta, 1/(2s))$-witness, we have that

$$\begin{aligned}
\texttt{cost}(\alpha) &= \sum_{U\subseteq S}\sum_{v\in U}\alpha(g_{U,v})(|U| - \frac{s}{2}) - \sum_{u,v\in S}\sum_{P\in\mathcal{P}(u,v)}\alpha(f_P^{uv})\left(\sum_{r\in V(P)} x_r\right)\\
&= \sum_{v\in S}\frac{s}{2}\alpha(g_{S,v}) - \sum_{u,v\in S}\alpha(f_{Q_{uv}}^{uv})\left(\sum_{r\in V(Q_{uv})} x_r\right)\\
&= \sum_{v\in S}\frac{s}{2}\frac{1}{4s} - \frac{1}{2s}\sum_{u,v\in S}\left(\sum_{r\in V(Q_{uv})} x_r\right)\\
&\leq \frac{s}{8} - \frac{1}{2s}\sum_{u,v\in S}\delta\\
&\leq \frac{s}{8} - \frac{\delta s}{2} = (\frac{1}{8} - \frac{\delta}{2})s.
\end{aligned}$$

This completes the proof. □

By the strong duality theorem and Lemma 4.2, we show that the existence of a certain fractional well-linkedness witness implies that a given function $x : V(G) \to \mathbb{Q}_0^+$ is not a feasible fractional solution of **Well-Linkedness LP**(G, w, t).

Lemma 4.3. *Let G be a graph, $w : V(G) \to \mathbb{Q}^+$, $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ and $t \in \mathbb{N}$. Let δ be such that there exists an $(s, \delta, 1/(2s))$-witness $(S, \mathcal{Q})$ with respect to (G, α). If $(\frac{1}{8} - \frac{\delta}{2})s > t$, then α cannot be extended to a feasible fractional solution of* **Well-Linkedness LP**(G, w, t).

Proof. Suppose that $(\frac{1}{8} - \frac{\delta}{2})s > t$. By Lemma 4.2, there exists a feasible fractional solution β' of **flow-LP**(G, S, α) with $\texttt{cost}(\beta') \geq (\frac{1}{8} - \frac{\delta}{2})s > t$. By Observation 4.2, this means that any feasible fractional solution β of **sep-LP**(G, S, α) satisfies $\texttt{cost}(\beta) > t$. In turn, by Observation 4.1, this means that α cannot be extended to a feasible fractional solution of **Well-Linkedness LP**(G, w, t). □

Next, we show that the existence of a "light" grid implies the existence of a certain fractional well-linkedness witness.

Lemma 4.4. *Let G be a graph, $x : V(G) \to \mathbb{Q}_0^+$ and $\delta \in \mathbb{Q}_0^+$. Let H be a $t \times t$-grid where $V(H) = \{h_{i,j} : i, j \in \{1, 2, \ldots, t\}\}$. Let φ be a minor model of H in G. Moreover,*

- *For $i \in \{1, 2, \ldots, t\}$, let $R_i = \bigcup_{j=1}^{t}\varphi(h_{i,j})$, and suppose that $\sum_{v\in V(R_i)} x_v < \delta/2$.*
- *For $j \in \{1, 2, \ldots, t\}$, let $C_j = \bigcup_{i=1}^{t}\varphi(h_{i,j})$, and suppose that $\sum_{v\in V(C_i)} x_v < \delta/2$.*

Then, there exists a $(t, \delta, 1/(2t))$-witness $(S, \mathcal{Q})$ with respect to (G, x) (where $|S| = t$).

Proof. For all $i \in \{1, 2, \ldots, t\}$, we arbitrarily choose some vertex in $\varphi(h_{i,i})$ and denote it by u_i. Denote $S = \{u_i : i \in \{1, 2, \ldots, t\}\}$. Intuitively, the set S contains one vertex from each set assigned to a vertex in the diagonal of H by φ. Note that $|S| = t$.

Since φ is a minor model of H in G, the following notation is well defined. For each pair of vertices $u_i, u_j \in U$, say $i < j$, we define a path $Q_{i,j}$ as follows. First, $Q_{i,j}$ contains (as a subpath) a path in G from u_i to some vertex in $\varphi(h_{i,j})$, say r, all of whose internal vertices belong to $\bigcup_{j'=i}^{j} \varphi(h_{i,j'})$. Then, $Q_{i,j}$ contains (as a subpath) a path in G from r to u_j, all of whose internal vertices belong to $\bigcup_{i'=i}^{j} \varphi(h_{i',j})$. Thus, $V(Q_{i,j}) \subseteq V(R_i) \cup V(C_j)$. In particular, this means that

$$\sum_{v \in V(Q_{i,j})} x_v \leq \sum_{v \in V(R_i)} x_v + \sum_{v \in V(C_i)} x_v < \delta/2 + \delta/2 = \delta.$$

Denote $\mathcal{Q} = \{Q_{i,j} : i, j \in \{1, 2, \ldots, t\}\}$. Now, to show that $(S, \mathcal{Q})$ is a $(t, \delta, 1/(2t))$-witness with respect to (G, S, x), it remains to show that for every vertex $r \in V(G)$, it holds that $(1/2|S|) \cdot |\{P \in \mathcal{Q} : r \in V(P)\}| \leq 1$. To this end, consider some vertex $r \in V(G)$. Let $i, j \in \{1, 2, \ldots, t\}$, $i \leq j$, denote the integers such that $r \in \varphi(h_{i,j})$ (if no such integers exist, then $|\{P \in \mathcal{Q} : r \in V(P)\}| = 0$). Note that $\{P \in \mathcal{Q} : r \in V(P)\} \subseteq \{Q_{i,j'} : j' \in \{i, i+1, \ldots, t\}\} \cup \{Q_{i',j} : i' \in \{1, 2, \ldots, j\}\}$, and therefore $|\{P \in \mathcal{Q} : r \in V(P)\}| \leq (t - i + 1) + j \leq 2t$. Thus, $(1/2|S|) \cdot |\{P \in \mathcal{Q} : r \in V(P)\}| \leq 1$. □

We are now ready to present the translation. Here, given a feasible fractional solution of **Well-Linkedness LP**, the translation entails the *division* of the value assigned to each variable x_v by t, along with the restriction of the result to the variable set of **Grid Hitting LP**. We also pay a minor penalty of multiplying t by a fixed constant.

Lemma 4.5. *There exists a fixed constant c such that for any triple (G, w, t) and feasible fractional solution α of* **Well-Linkedness LP**(G, w, t)*, the following holds. Define $\alpha' : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ by $\alpha'(x_v) = (1/t) \cdot \alpha(x_v)$ for all $v \in V(G)$. Then, α' is a feasible fractional solution of* **Grid Hitting LP**(G, w, ct) *such that $\texttt{cost}(\alpha') = (1/t) \cdot \texttt{cost}(\alpha)$.*

Proof. Let c and d be fixed constants determined later. Suppose, by way of contradiction, that α' is not a feasible fractional solution of **Grid Hitting LP**(G, w, ct). Then, there exists a set $S \in \texttt{Grid}_{ct}(G)$ such that $\sum_{v \in S} \alpha'(x_v) < 1$. By the definition of α', this means that $\sum_{v \in S} \alpha(x_v) < t$.

Let H be a $ct \times ct$-grid with $V(H) = \{h_{i,j} : i, j \in \{1, 2, \ldots, ct\}\}$ and $E(H) = \{\{h_{i,j}, h_{i',j'}\} : |i - i'| + |j - j'| = 1\}$. Because $S \in \texttt{Grid}_{ct}(G)$, there exists a minor model $\varphi : V(H) \to 2^S$ of H in $G[S]$. For every $i \in \{1, 2, \ldots, ct\}$, denote $R_i = \bigcup_{j=1}^{ct} \varphi(h_{i,j})$. Similarly, for every $j \in \{1, 2, \ldots, ct\}$, let $C_j = \bigcup_{i=1}^{ct} \varphi(h_{i,j})$. Intuitively, for all $i \in \{1, 2, \ldots, ct\}$, the set R_i captures a "row" in the image of H in G, and for all $j \in \{1, 2, \ldots, ct\}$, the set C_j captures a "column" in the image of H in G.

Let $\overline{I}$ denote the set of integers $i \in \{1, 2, \ldots, ct\}$ such that $\sum_{v \in R_i} \alpha(x_v) \geq 2/c$. Similarly, let $\overline{J}$ denote the set of integers $j \in \{1, 2, \ldots, ct\}$ such that $\sum_{v \in C_j} \alpha(x_v) \geq 2/(cd)$. We claim that since $\sum_{v \in S} \alpha(x_v) < t/d$, it holds that $|\overline{I}| < ct/2$. To see this, suppose by way of contradiction that this claim is false. Then, $\sum_{v \in S} \alpha(x_v) \geq \sum_{i=1}^{ct} \sum_{v \in R_i} \alpha(x_v) \geq ct/2 \cdot 2/(cd) = t/d$, which is a contradiction. Symmetrically, it holds that $|\overline{J}| < ct/2$. Denote $I = \{1, 2, \ldots, ct\} \setminus \overline{I}$ and $J = \overline{J}$. Then, $|I| > ct/2$ and $|J| > ct/2$. Let us denote $V_I = \bigcup_{i \in I} R_i$ and $V_J = \bigcup_{j \in J} C_j$.

Let $H^\star$ be a $(ct/2) \times (ct/2)$-grid with $V(H^\star) = \{h^\star_{i,j} : i, j \in \{1, 2, \ldots, ct/2\}\}$ and $E(H^\star) = \{\{h^\star_{i,j}, h^\star_{i',j'}\} : |i - i'| + |j - j'| = 1\}$. For all $i \in \{1, 2, \ldots, ct/2\}$, let $I[i]$ denote the i-th smallest integer in I, and for all $j \in \{1, 2, \ldots, ct/2\}$, let $J[j]$ denote the j-th smallest integer in J. Define $\varphi^\star : V(H^\star) \to (V_I \cup V_J)$ as follows. For all $h^\star_{i,j} \in V(H^\star)$, define $\varphi^\star(h^\star_{i,j}) = \bigcup(\{\varphi(h_{I[i],a}) : a \in \{J[j], J[j]+1, \ldots, J[j+1]-1\}\} \cup \{\varphi(h_{a,J[j]}) : a \in \{I[i], I[i]+1, \ldots, I[i+1]-1\}\})$. Observe that $\varphi^\star$ is a minor model of $H^\star$ in G.

Denote $\delta = 4/c$. By Lemma 4.4, we obtain that $G[V_I \cup V_J]$ contains a $(ct/2, \delta, 1/(ct/2))$-witness $(S, \mathcal{Q})$ with respect to (G, α). However, by Lemma 4.3, if $(\frac{1}{8} - \frac{\delta}{2})|S| > t$, then α is not a feasible fractional solution of **Well-Linkedness LP**(G, w, t), which is a contradiction. Note that $(\frac{1}{8} - \frac{\delta}{2})|S| = (\frac{1}{8} - \frac{2}{c})\frac{ct}{2} = (\frac{c}{16} - 1)t$. Thus, by choosing $c = 100$ we reach the desired contradiction.

So far, we derived that α' is a feasible fractional solution of **Grid Hitting LP**(G, w, ct). Observe that

$$\texttt{cost}(\alpha') = \sum_{v \in V(G)} w_v \alpha'(x_v) = (1/t) \cdot \sum_{v \in V(G)} w_v \alpha(x_v) = (1/t) \cdot \texttt{cost}(\alpha).$$

This completes the proof. □

5 From Grid Hitting LP to Pairwise-Flow Hitting LP

In this section, we translate a feasible fractional solution of **Grid Hitting LP** to a feasible fractional solution of a new LP, called **Pairwise-Flow LP**.

5.1 (h, t)-Pairwise Flow and the Pairwise-Flow Hitting LP

For a graph G and an integer h, we let $\texttt{ConPart}(G, h)$ denote the set of all tuples $(X_1, X_2, \ldots, X_h)$ for which there exists $X \subseteq V(G)$ such that $X = X_1 \uplus X_2 \uplus \cdots \uplus X_h$, and X_i is a connected set for all $i \in \{1, 2, \ldots, h\}$. Having this notation, we define the following notion.

Definition 5.1. *A graph G has an (h, t)-*pairwise flow *if there exists $\overline{X} \in \texttt{ConPart}(G, h)$ such that for all distinct $i, j \in \{1, 2, \ldots, h\}$, the maximum number of pairwise vertex-disjoint paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$ is lower bounded by t. The maximum integer t such that G has an (h, t)-pairwise flow is denoted by $\texttt{pf}_h(G)$.*

In **Pairwise-Flow Hitting LP**, we are given a graph G that is H minor-free for a graph H with $|V(H)| = h$, a weight function $w : V(G) \to \mathbb{Q}^+$ and $t \in \mathbb{N}$. Our objective is to delete vertices of minimum weight so that G will not have an $(h, t+1)$-pairwise flow. Each variable x_v indicates whether we delete v. For every $\overline{X} \in \texttt{ConPart}(G, h)$ and $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, we have variables $\lambda^{\overline{X},(i,j)}$ and $y_v^{\overline{X},(i,j)}$ for all $v \in V(G)$. Informally, setting the variable $\lambda^{\overline{X},(i,j)}$ to 1 indicates that the maximum number of pairwise vertex-disjoint paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$, after the deletion of vertices as indicated by the variables x_v for all $v \in V(G)$, should be upper bounded by t. In turn, the satisfaction of this indication is realized by using the variables $y_v^{\overline{X},(i,j)}$ for all $v \in V(G)$, which specify (by setting their value to 1) which vertices hit, together with the vertices already deleted, all paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$.[6]

[6]We remark that **Pairwise-Flow Hitting LP**(G, w, h, t) "approximates" the notion of (h, t)-pairwise flow, but does not captures it "exactly", in the following sense. If $S \subseteq V(G)$ is a subset such that $G - S$ does not have $(h, t+1)$-pairwise flow, by setting the variables x_v, $v \in S$, to 1, and x_v, $v \notin S$, to 0, we might not be able to set the rest of the variables so that we obtain a feasible solution. However, our later arguments will imply that there exists t' close to t, such that if $S \subseteq V(G)$ is a subset such that $G - S$ does not have $(h, t'+1)$-pairwise flow, then setting variables as above gives rise to a feasible solution. We can add constraints to make the LP capture the measure precisely, but this only complicates it unnecessarily.

Pairwise-Flow Hitting LP(G, w, h, t)**:**

$$\begin{array}{lll} \min & \displaystyle\sum_{v \in V(G)} w_v x_v & \\ \text{s.t.} & \displaystyle\sum_{i=1}^{h-1} \sum_{j=i+1}^{h} \lambda^{\overline{X},(i,j)} \geq 1 & \forall \overline{X} \in \texttt{ConPart}(G, h) \\ & \displaystyle\sum_{v \in V(G)} y_v^{\overline{X},(i,j)} \leq t & \forall \overline{X} \in \texttt{ConPart}(G, h), i, j \in \{1, \ldots, h\}, i < j \\ & \displaystyle\sum_{v \in V(P)} (x_v + y_v^{\overline{X},(i,j)}) \geq \lambda^{\overline{X},(i,j)} & \forall \overline{X} \in \texttt{ConPart}(G, h), i, j \in \{1, \ldots, h\}, i < j, \\ & & P \in \mathcal{P}_{G-(X \setminus (\overline{X}[i] \cup \overline{X}[j]))}(\overline{X}[i], \overline{X}[j]) \\ & x_v \geq 0, y_v^{\overline{X},(i,j)} \geq 0, \lambda^{\overline{X},(i,j)} \geq 0 & \end{array}$$

The rationale behind the definition of this LP, in the context of our work, stems from the following result.

Lemma 5.1. *Let H be a graph and denote $h = |V(H)|$. For any graph G that is H-minor free, it holds that $\texttt{pf}_h(G) < 5\binom{h}{2}^4(\texttt{tw}(G) + 1)$.*

Towards the proof of Lemma 5.1, we first establish the following simple result.

Lemma 5.2. *Let G be a graph of treewidth w. Let $\mathcal{P}$ be a collection of t pairwise internally vertex-disjoint paths in G between two vertices $u, v \in V(G)$. Then, for any $k, \ell \in \mathbb{N}$ such that $t \geq k(3w + 3 + 2\ell)$, there exists a subset $B \subseteq V(G)$ of size at most $k(w+1)$ such that the set of connected components of $G - B$ can be partitioned into k (pairwise disjoint) sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$, with the property that for all $i \in \{1, 2, \ldots, k\}$, the graph $G[V(\mathcal{C}_i) \cup \{u, v\}]$ has at least ℓ paths from $\mathcal{P}$.*

Proof. We prove the lemma by induction on k. In the basis, when $k = 0$, the claim (of the lemma) is vacuously true. Now, consider some $k \geq 1$, and suppose that the claim is true for $k - 1$.

Since $\texttt{tw}(G) = w$, it holds that G has tree decomposition (T, β) of width w. For all $b \in V(T)$, it holds that $|\beta(b)| \leq w + 1$. Without loss of generality, suppose that T is binary (otherwise, we can construct another tree decomposition of the same width where the tree is binary). Let x be a vertex in T of maximum distance from the root of T such that $G[(\gamma(x) \setminus \beta(x)) \cup \{u, v\}]$ has at least ℓ paths from $\mathcal{P}$. (Since $|\mathcal{P}| \geq k(3w + 3 + 2\ell)$, $|\beta(r)| \leq w + 1$ for the root r of T and the paths in $\mathcal{P}$ are internally vertex-disjoint, the graph $G[(\gamma(r) \setminus \beta(r)) \cup \{u, v\}]$ has this property, and hence such x exists.)

Denote $B' = \beta(x)$, and let $\mathcal{C}_k$ be the set of connected components of $G - B'$ that belong to $G[\gamma(x)] - B'$. Then, the graph $G[V(\mathcal{C}_k) \cup \{u, v\}]$ has at least ℓ paths from $\mathcal{P}$. By our choice of x, it holds that for any child y of x in T (if such a child exists), $G[(\gamma(y) \setminus \beta(y)) \cup \{u, v\}]$ has less than ℓ paths from $\mathcal{P}$ and $|\beta(y)| \leq w + 1$. Since the paths in $\mathcal{P}$ are internally vertex-disjoint, this means that $G[V(\mathcal{C}_k) \cup \{u, v\}]$ has at most $2(\ell + w + 1)$ paths from $\mathcal{P}$. Moreover, at most $w + 1$ paths in $\mathcal{P}$ include at least one vertex from $B' \setminus \{u, v\}$. Therefore, if we remove from $\mathcal{P}$ all paths that contain at least one vertex from $(B' \cup V(\mathcal{C}_k)) \setminus \{u, v\}$, we are left with at least $(k-1)(3w + 3 + 2\ell)$ paths. Thus, by the inductive hypothesis, there exists a subset $B'' \subseteq V(G)$ of size at most $(k-1)(w+1)$ such that the set of connected components of $G - (\gamma(x) \cup B'')$ can be partitioned into $k - 1$ (pairwise disjoint) sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{k-1}$, with the property that for all $i \in \{1, 2, \ldots, k-1\}$, the graph $G[V(\mathcal{C}_i) \cup \{u, v\}]$ has at least ℓ paths from $\mathcal{P}$.

Now, denote $B = B' \cup B''$. Then, $|B| \le (w+1) + (k-1)(w+1) = k(w+1)$. Moreover, $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$ is a partition of the set of connected components of $G - B$ with the property that for all $i \in \{1, 2, \dots, k\}$, the graph $G[V(\mathcal{C}_i) \cup \{u, v\}]$ has at least ℓ paths from $\mathcal{P}$. This completes the proof. $\square$

We need to extend Lemma 5.2 as follows.

Lemma 5.3. *Let G be a graph of treewidth w. Let $U = \{u_1, u_2, \dots, u_h\} \subseteq V(G)$ be a set of h vertices such that for all distinct $u_i, u_j \in U$, there exist t pairwise internally vertex-disjoint paths in $G - (U \setminus \{u_i, u_j\})$ between u_i and u_j. Suppose that $t \ge 5\binom{h}{2}^3(w+1)$. Then, there exists a subset $B \subseteq V(G)$ of size at most $\binom{h}{2}^2(w+1)$ such that the set $\mathcal{C}$ of connected components of $G - B$ has the property that for all distinct $i, j \in \{1, 2, \dots, h\}$ there exist at least $\binom{h}{2}$ components $C \in \mathcal{C}$ such that $G[V(C) \cup \{u_i, u_j\}]$ has at least one path between u_i and u_j that does not include any vertex from $U \setminus \{u_i, u_j\}$.*

Proof. Let $I = \{(i, j) : i < j, i, j \in \{1, 2, \dots, h\}\}$. For each pair $(i, j) \in I$, let $\mathcal{P}_{i,j}$ denote a set of t pairwise internally vertex-disjoint paths in $G - (U \setminus \{u_i, u_j\})$ between u_i and u_j. By Lemma 5.2 with $k = \binom{h}{2}$ and $\ell = \binom{h}{2}^2(w+1)$, and since $t \ge 5\binom{h}{2}^3(w+1) \ge \binom{h}{2}(3w+3+2\binom{h}{2}^2(w+1)) = k(3w+3+2\ell)$, the following claim holds. For each pair $(i, j) \in I$, there exists a subset $B_{i,j} \subseteq V(G)$ of size at most $\binom{h}{2}(w+1)$ such that the set of connected components of $G - B_{i,j}$ can be partitioned into $\binom{h}{2}$ (pairwise disjoint) sets $\mathcal{C}^{i,j}_1, \mathcal{C}^{i,j}_2, \dots, \mathcal{C}^{i,j}_{\binom{h}{2}}$, with the property that for all $q \in \{1, 2, \dots, \binom{h}{2}\}$, the graph $G[V(\mathcal{C}^{i,j}_q) \cup \{u, v\}]$ has at least $\binom{h}{2}^2(w+1)$ paths from $\mathcal{P}_{i,j}$.

Denote $B = \bigcup_{(i,j) \in I} B_{i,j}$, and let $\mathcal{C}$ be the set of connected components of $G - B$. Then, $|B| \le \binom{h}{2}^2(w+1)$. Now, let $\mathcal{D}$ denote the set of connected components of $G - B$. Consider some pair $(i, j) \in I$. Then, $|B \setminus B_{i,j}| \le (\binom{h}{2}^2 - 1)(w+1)$, and therefore for all $q \in \{1, 2, \dots, \binom{h}{2}\}$, the graph $G[(V(\mathcal{C}^{i,j}_q) \setminus B) \cup \{u, v\}]$ has at least $\binom{h}{2}^2(w+1) - |B \setminus B_{i,j}| \ge 1$ paths from $\mathcal{P}_{i,j}$. For all $q \in \{1, 2, \dots, \binom{h}{2}\}$, the graph $G[(V(\mathcal{C}^{i,j}_q) \setminus B) \cup \{u, v\}]$ is a subgraph of G induced by $\{u, v\}$ and a distinct collection of components from $\mathcal{C}$. However, this means that $\mathcal{C}$ has at least $\binom{h}{2}$ components $C \in \mathcal{C}$ such that $G[V(C) \cup \{u_i, u_j\}]$ has at least one path from $\mathcal{P}_{i,j}$ (which is a path between u_i and u_j that does not include any vertex from $U \setminus \{u_i, u_j\}$). Since the choice of $(i, j) \in I$ was arbitrary, the proof is complete. $\square$

Before we turn prove Lemma 5.1, we need to state one more lemma, based on Lemma 5.3. Later, we will actually directly use this lemma rather than Lemma 5.1, thus we need this lemma exactly in the form below.

Lemma 5.4. *Let H be a graph and denote $h = |V(H)|$. Let G be a graph that is H-minor free, and let $\overline{X} \in \texttt{ConPart}(G, h)$ be such that for all distinct $i, j \in \{1, 2, \dots, h\}$, there exist a t pairwise vertex-disjoint paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$. Then, $t < 5\binom{h}{2}^4(\texttt{tw}(G - X) + 1)$.*

Proof. Suppose, by way of contradiction, that $t \ge 5\binom{h}{2}^4(\texttt{tw}(G-X)+1)$. Because G is H-minor free, it does not have K_h (the clique on h vertices) as a minor. Let G' be the graph obtained from G by subdividing once every edge $\{u, v\} \in E(G)$ for which there is no $i \in \{1, 2, \dots, h\}$ such that $u, v \in \overline{X}[i]$. Note that G' also does not have K_h as a minor, and it has the same treewidth as G. Recall that for all $i \in \{1, 2, \dots, h\}$, we have that $\overline{X}[i]$ is a connected set in G and hence also in G', and that for all distinct $i, j \in \{1, 2, \dots, h\}$, we have that $\overline{X}[i] \cap \overline{X}[j] = \emptyset$. Thus, it is well defined to contract in G' each set $\overline{X}[i]$ into a new single vertex, which we denote by u_i. We let $G^\star$ denote the resulting graph, that is, $G^\star = (((G'/\overline{X}[1])/\overline{X}[2])/\cdots)/\overline{X}[h]$. Moreover, denote $U = \{u_i : i \in \{1, 2, \ldots, h\}\}$. Since $G^\star$ is a minor of G', we have that $G^\star$ is K_h-minor free, and $\mathtt{tw}(G^\star) \leq \mathtt{tw}(G')$. Moreover, for all distinct $i, j \in \{1, 2, \ldots, h\}$, there exist t pairwise vertex-disjoint paths in $G' - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$ and at least one internal vertex not in X. Thus, for all distinct $i, j \in \{1, 2, \ldots, h\}$, there exist t pairwise internally vertex-disjoint paths in $G^\star - (U \setminus \{u_i, u_j\})$ between u_i and u_j.

Denote $w = \mathtt{tw}(G^\star)$. Observe that $G^\star - U$ is a minor of $G' - X$, and therefore $\mathtt{tw}(G^\star - U) \leq \mathtt{tw}(G' - X)$. Moreover, as $|U| = \binom{h}{2}$, we have that $w \leq \mathtt{tw}(G^\star - U) + \binom{h}{2}$. Therefore, $w \leq \mathtt{tw}(G' - X) + \binom{h}{2} = \mathtt{tw}(G - X) + \binom{h}{2}$, which implies that $5\binom{h}{2}^3(w+1) \leq 5\binom{h}{2}^3(w + \binom{h}{2} + 1) \leq 5\binom{h}{2}^4(w+1) \leq t$. By Lemma 5.3, there exists a subset $B \subseteq V(G^\star)$ of size at most $\binom{h}{2}^2(w+1)$ such that for all distinct $u_i, u_j \in U$, the graph $G^\star - B$ has at least $\binom{h}{2}$ connected components C with the property that $G^\star[V(C) \cup \{u_i, u_j\}]$ has at least one path between u_i and u_j that does not include any vertex from $U \setminus \{u_i, u_j\}$. This means that for each pair of distinct vertices $u_i, u_j \in U$, we can choose a unique component $C_{\{i,j\}}$ in $G^\star - B$ with the property that $G^\star[V(C_{\{i,j\}}) \cup \{u_i, u_j\}]$ has at least one path between u_i and u_j that does not include any vertex from $U \setminus \{u_i, u_j\}$. However, this means that $G^\star$ has K_h as a minor (in fact, it even has a subdivision of K_h as a subgraph). Thus, we have reached a contradiction. □

We are now ready to prove Lemma 5.1.

Proof of Lemma 5.1. Let G be a graph that is H-minor free, and denote $t = \mathtt{pf}_h(G)$. By the definition of $\mathtt{pf}_h(G)$, there exists $\overline{X} \in \mathtt{ConPart}(G, h)$ such that for all distinct $i, j \in \{1, 2, \ldots, h\}$, there exist t pairwise vertex-disjoint paths in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ with one endpoint in $\overline{X}[i]$ and the other endpoint in $\overline{X}[j]$. By Lemma 5.4, $t < 5\binom{h}{2}^4(\mathtt{tw}(G - X) + 1)$. Since $\mathtt{tw}(G - X) \leq \mathtt{tw}(G)$, the proof is complete. □

5.2 Duality of Pairwise-Flow Hitting LP

Towards the translation of a feasible fractional solution of **Grid Hitting LP** to a feasible fractional solution of **Pairwise-Flow Hitting LP**, we need to establish several claims. We begin by analyzing an LP called **Penalized Flow Hitting LP**, which can roughly be viewed as program nested in **Pairwise-Flow Hitting LP**. In **Penalized Flow Hitting LP**, we are given a graph G, a penalty function $x : V(G) \to \mathbb{Q}_0^+$, and two disjoint subsets $A, B \subseteq V(G)$. We stress that x_v, for any $v \in V(G)$, is not a variable. Roughly speaking, the objective of the LP can be viewed as follows. Each path $P \in \mathcal{P}(A, B)$ is already "partially hit"—specifically, it is hit with $\sum_{v \in V(P)} x_v$. Then, our objective is to choose a minimum number of vertices so that every path $P \in \mathcal{P}(A, B)$ is "fully hit"—that is, it is hit with 1. Each variable y_v indicates whether the vertex v is chosen for this purpose.

Penalized Flow Hitting LP(G, x, A, B)**:**

$$\min \sum_{v \in V(G)} y_v$$

$$\text{s.t.} \sum_{v \in V(P)} y_v \geq 1 - \sum_{v \in V(P)} x_v \qquad \forall P \in \mathcal{P}(A, B)$$

$$y_v \geq 0$$

First, we relate this LP to **Pairwise-Flow Hitting LP** as follows. We remark that the second item in this lemma will be required only in a later section (Section 6.2).

Lemma 5.5. *Let G be an H-minor free graph with $h = |V(H)|$, $w : V(G) \to \mathbb{Q}^+$, $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ and $t \in \mathbb{N}$.*

- *On the one hand, if α cannot be extended to a feasible fractional solution of* **Pairwise-Flow Hitting LP**(G, w, h, t)*, then there exists* $\overline{X} \in \texttt{ConPart}(G, h)$ *such that for all* $i, j \in \{1, 2, \ldots, h\}$ *with* $i < j$*, the optimum of* **Penalized Flow Hitting LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), \alpha, \overline{X}[i], \overline{X}[j])$ *is larger than* t*.*

- *On the one hand, if there exists* $\overline{X} \in \texttt{ConPart}(G, h)$ *such that for all* $i, j \in \{1, 2, \ldots, h\}$ *with* $i < j$*, the optimum of* **Penalized Flow Hitting LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), \alpha, \overline{X}[i], \overline{X}[j])$ *is larger than* $t' = \dfrac{h(h-1)}{2} \cdot t$*, then* α *cannot be extended to a feasible fractional solution of* **Pairwise-Flow Hitting LP**(G, w, h, t)*.*

Proof. Given $\overline{X} \in \texttt{ConPart}(G, h)$, denote $B^{\overline{X}} = \{\lambda^{\overline{X},(i,j)} : i, j \in \{1, \ldots, h\}, i < j\} \cup \{y_v^{\overline{X},(i,j)} : i, j \in \{1, \ldots, h\}, i < j, v \in V(G)\}$. We say that an assignment $\beta : B \to \mathbb{Q}_0^+$ is *good for* $\overline{X}$ if it satisfies the following constraints:

- $\displaystyle\sum_{i=1}^{h-1} \sum_{j=i+1}^{h} \beta(\lambda^{\overline{X},(i,j)}) \geq 1.$

- $\displaystyle\sum_{v \in V(P)} (\alpha(x_v) + \beta(y_v^{\overline{X},(i,j)})) \geq \beta(\lambda^{\overline{X},(i,j)})$ for all $i, j \in \{1, \ldots, h\}$ with $i < j$ and $P \in \mathcal{P}_{G-(X \setminus (\overline{X}[i] \cup \overline{X}[j]))}(\overline{X}[i], \overline{X}[j])$.

Note that there does not exist a feasible fractional solution α' of **Pairwise-Flow Hitting LP**(G, w, h, t) that extends α if and only if there exists $\overline{X} \in \texttt{ConPart}(G, h)$ such that every assignment β that is good for $\overline{X}$ satisfies $\sum_{v \in V(G)} \beta(y_v^{\overline{X},(i,j)}) > t$.

On the one hand, suppose that there does not exist a feasible fractional solution α' of **Pairwise-Flow Hitting LP**(G, w, h, t) that extends α. Then, every assignment β that is good for $\overline{X}$ satisfies $\sum_{v \in V(G)} \beta(y_v^{\overline{X},(i,j)}) > t$ In particular, this claim holds for every assignment β that is good for $\overline{X}$ and such that $\beta(\lambda_v^{i,j}) = 1$ for some $i, j \in \{1, \ldots, h\}$ with $i < j$ and $\beta(\lambda_v^{i,j}) = 0$ for all $i', j' \in \{1, \ldots, h\}$ with $'i < j'$ and $(i, j) \neq (i', j')$. However, this means that or all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the optimum of **Penalized Flow Hitting LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), w, \alpha, \overline{X}[i], \overline{X}[j])$ is larger than t.

On the other hand, suppose that there exists $\overline{X} \in \texttt{ConPart}(G, h)$ such that for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the optimum of **Penalized Flow Hitting LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), \alpha, \overline{X}[i], \overline{X}[j])$ is larger than $t' = \dfrac{h(h-1)}{2} \cdot t$. Targeting a contradiction, suppose that there exists a feasible fractional solution α' of **Pairwise-Flow Hitting LP**(G, w, h, t) that extends α. Then, there exists an assignment β that is good for $\overline{X}$ and satisfies $\sum_{v \in V(G)} \beta(y_v^{\overline{X},(i,j)}) \leq t$. Let $i^\star, j^\star \in \{1, 2, \ldots, h\}$ with $i^\star < j^\star$ be such that $\beta(\lambda^{\overline{X},(i^\star,j^\star)}) \geq 1/\binom{h}{2} = \frac{2}{h(h-1)}$. (Because $\displaystyle\sum_{i=1}^{h-1} \sum_{j=i+1}^{h} \beta(\lambda^{\overline{X},(i,j)}) \geq 1$, such a pair $i^\star, j^\star$ exists.) Now, observe that $\displaystyle\sum_{v \in V(P)} (\alpha(x_v) + \beta(y_v^{\overline{X},(i^\star,j^\star)})) \geq \frac{2}{h(h-1)}$ for every $P \in \mathcal{P}_{G-(X \setminus (\overline{X}[i] \cup \overline{X}[j]))}(\overline{X}[i], \overline{X}[j])$. Accordingly, we define $\gamma : \{y_v^{\overline{X},(i,j)} : i, j \in \{1, \ldots, h\}, i < j, v \in V(G)\} \to \mathbb{Q}_0^+$ as follows: For every $y_v^{\overline{X},(i,j)}$ in the domain, define $\gamma(y_v^{\overline{X},(i,j)}) = \beta(y_v^{\overline{X},(i,j)})/(\dfrac{2}{h(h-1)})$. Then, the two following properties hold:

- $\displaystyle\sum_{v \in V(G)} \beta(y_v^{\overline{X},(i,j)}) \leq t'.$

- $\sum_{v \in V(P)} (\alpha(x_v) + \beta(y_v^{\overline{X},(i^\star,j^\star)})) \geq 1$ for every $P \in \mathcal{P}_{G-(X\setminus(\overline{X}[i]\cup\overline{X}[j]))}(\overline{X}[i], \overline{X}[j])$.

However, this implies that the optimum of **Penalized Flow Hitting LP**$(G-(X\setminus(\overline{X}[i] \cup \overline{X}[j])), \alpha, \overline{X}[i], \overline{X}[j])$ is at most t', and hence we have reached a contradiction. $\square$

Now, to analyze **Penalized Flow Hitting LP**, we need to consider its dual, namely, **Penalized Flow Packing LP**.

Penalized Flow Packing LP(G, x, A, B)**:**

$$\begin{array}{lll} \max & \sum_{P \in \mathcal{P}(A,B)} z_P(1 - \sum_{v \in V(P)} x_v) & \\ \text{s.t.} & \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} z_P \leq 1 & \forall v \in V(G) \\ & z_P \geq 0 & \end{array}$$

From Proposition 3.2, we obtain the following observation.

Observation 5.1. *Let G be a graph, $x : V(G) \to \mathbb{Q}_0^+$, and $A, B \subseteq V(G)$ be disjoint. Then, the optimum of* **Penalized Flow Hitting LP**(G, x, A, B) *is equal to the optimum of* **Penalized Flow Packing LP**(G, x, A, B).

5.3 Translation

Towards the translation, we analyze **Penalized Flow Packing LP**. To this end, it is convenient to rephrase **Penalized Flow Packing LP** in the form a well-known LP, namely, **Min-Cost Circulation LP**. Here, we are given a *directed* graph D, a capacity function $c : A(D) \to \mathbb{N}$, and a cost function $k : A(D) \to \mathbb{Q}$. For every arc $(u, v) \in A(D)$, we have a variable $f_{(u,v)}$ that specifies the amount of flow to traverse (u, v). The first set of constraints encodes the flow-preservation requirement (we do not have any source or sink), and the second set of constraints ensures that the flow respects the capacities of the arcs.

Min-Cost Circulation LP(D, c, k)**:**

$$\begin{array}{lll} \min & \sum_{(u,v) \in A(D)} k_{(u,v)} f_{(u,v)} & \\ \text{s.t.} & \sum_{u \in N_{\text{in}}(v)} f_{(u,v)} - \sum_{u \in N_{\text{out}}(v)} f_{(v,u)} = 0 & \forall v \in V(D) \\ & f_{(u,v)} \leq c_{(u,v)} & \forall (u,v) \in A(D) \\ & f_{(u,v)} \geq 0 & \end{array}$$

To rephrase **Penalized Flow Packing LP**, we will make use of the following notation. For a path $P \in \mathcal{P}(A, B)$, we orient P from its endpoint in A to its endpoint in B. Accordingly, for an edge an edge $\{u, v\} \in E(G)$, we write $(u, v) \in A(P)$ if $\{u, v\} \in E(P)$ and u is closer than v in P to the endpoint of P in A.

Let us now show that we can indeed rephrase **Penalized Flow Packing LP** as **Min-Cost Circulation LP**. To this end, for a quadruple (G, x, A, B), we define a triple $\texttt{circulate}(G, x, A, B) = (D, c, k)$ as follows. First, set $V(D) = \{s, t\} \cup \{v_{\text{in}} : v \in V(G)\} \cup \{v_{\text{out}} : v \in V(G)\}$ and $A(D) = \{(t, s)\} \cup \{(s, v_{\text{in}}) : v \in A\} \cup \{(v_{\text{out}}, t) : v \in B\} \cup \{(v_{\text{in}}, v_{\text{out}}) : v \in V(G)\} \cup \{(u_{\text{out}}, v_{\text{in}}) :$

$\{u, v\} \in E(G)\}$. Next, define $c : A(D) \to \mathbb{N}$ as follows. Assign $c((t, s)) = |V(G)|$, and for all of the remaining arcs in $A(D)$ assign capacity 1. Lastly, define $k : A(D) \to \mathbb{Q}$ as follows. Assign $k((t, s)) = -1$. Then, for all $v \in V(G)$, assign $k((v_{\text{in}}, v_{\text{out}})) = x_v$, and for all of the remaining arcs in $A(D)$ assign cost 0.

Lemma 5.6. *Given a feasible fractional solution α of* **Penalized Flow Packing LP**(G, x, A, B)*, define $\beta : \{f_{(u,v)} : (u, v) \in A(D)\} \to \mathbb{Q}_0^+$ as follows.*

- $\beta(f_{(t,s)}) = \sum_{P \in \mathcal{P}(A,B)} \alpha(z_P)$.
- *For all $v \in A$, define* $\beta(f_{(s,v_{\text{in}})}) = \sum_{P \in \mathcal{P}(\{v\},B)} \alpha(z_P)$.
- *For all $v \in B$, define* $\beta(f_{(v_{\text{out}},t)}) = \sum_{P \in \mathcal{P}(A,\{v\})} \alpha(z_P)$.
- *For all $v \in V(G)$, define* $\beta(f_{(v_{\text{in}},v_{\text{out}})}) = \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P)$.
- *For all $\{u, v\} \in E(G)$, define* $\beta(f_{(u_{\text{out}},v_{\text{in}})}) = \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } (u,v) \in A(P)}} \alpha(z_P)$.

Then, β is a feasible fractional solution of **Min-Cost Circulation LP**(D, c, k) *whose cost is $-\mathtt{cost}(\alpha)$, where $(D, c, k) = \mathtt{circulate}(G, x, A, B)$.*

Proof. To show that β is a feasible fractional solution of **Min-Cost Circulation LP**(D, c, k), let us first show that $\sum_{u \in N_{\text{in}}(v)} \beta(f_{(u,v)}) = \sum_{u \in N_{\text{out}}(v)} \beta(f_{(v,u)})$ for all $v \in V(D)$. We verify this claim as follows. First, for the vertex s, we have that

$$\begin{aligned} \sum_{u \in N_{\text{in}}(s)} \beta(f_{(u,s)}) &= \beta(f_{(t,s)}) = \sum_{P \in \mathcal{P}(A,B)} \alpha(z_P) \\ &= \sum_{v \in A} \sum_{P \in \mathcal{P}(\{v\},B)} \alpha(z_P) = \sum_{v \in A} \beta(f_{(s,v_{\text{in}})}) = \sum_{u \in N_{\text{out}}(s)} \beta(f_{(s,u)}). \end{aligned}$$

For the vertex t, we have that

$$\begin{aligned} \sum_{u \in N_{\text{out}}(t)} \beta(f_{(t,u)}) &= \beta(f_{(t,s)}) = \sum_{P \in \mathcal{P}(A,B)} \alpha(z_P) \\ &= \sum_{v \in B} \sum_{P \in \mathcal{P}(A,\{v\})} \alpha(z_P) = \sum_{v \in B} \beta(f_{(v_{\text{out}},t)}) = \sum_{u \in N_{\text{in}}(t)} \beta(f_{(u,t)}). \end{aligned}$$

For any vertex $v_{\text{in}} \in V(D)$, we have that

$$\begin{aligned} \sum_{u \in N_{\text{out}}(v_{\text{in}})} \beta(f_{(v_{\text{in}},u)}) &= \beta(f_{(v_{\text{in}},v_{\text{out}})}) = \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \\ &= \sum_{u \in N_G(v)} \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } (u,v) \in A(P)}} \alpha(z_P) + \sum_{\substack{P \in \mathcal{P}(\{v\},B) \\ \text{if } v \in A}} \alpha(z_P) \\ &= \sum_{u \in N_{\text{in}}(v_{\text{in}}) \setminus \{s\}} \beta(f_{(u,v_{\text{in}})}) + \beta(f_{(s,v_{\text{in}})}) = \sum_{u \in N_{\text{in}}(v_{\text{in}})} \beta(f_{(u,v_{\text{in}})}). \end{aligned}$$

Similarly, for any vertex $v_{\text{out}} \in V(D)$, we have that

$$\begin{aligned}
\sum_{u \in N_{\text{in}}(v_{\text{out}})} \beta(f_{(u,v_{\text{out}})}) &= \beta(f_{(v_{\text{in}},v_{\text{out}})}) = \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \\
&= \sum_{u \in N_G(v)} \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } (v,u) \in A(P)}} \alpha(z_P) + \sum_{\substack{P \in \mathcal{P}(A,\{v\}) \\ \text{if } v \in B}} \alpha(z_P) \\
&= \sum_{u \in N_{\text{out}}(v_{\text{out}}) \setminus \{t\}} \beta(f_{(v_{\text{out}},u)}) + \beta(f_{(v_{\text{out}},t)}) = \sum_{u \in N_{\text{out}}(v_{\text{out}})} \beta(f_{(v_{\text{out}},u)}).
\end{aligned}$$

In addition, we need to show that $\beta(f_{(u,v)}) \leq c_{(u,v)}$ for all $(u,v) \in A(D)$. Here, we rely on the supposition that α is a feasible fractional solution of **Penalized Flow Packing LP**(G,x,A,B). First, for all $(v_{\text{in}}, v_{\text{out}}) \in A(D)$, we have that $\beta(f_{(v_{\text{in}},v_{\text{out}})}) = \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \leq 1 = c_{(v_{\text{in}},v_{\text{out}})}$. For all $(u, v_{\text{in}}) \in A(D)$ (where possibly $u = s$), we have that $\beta(f_{(u,v_{\text{in}})}) \leq \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \leq 1 \leq c_{(u,v_{\text{out}})}$. Similarly, for all $(v_{\text{out}}, u) \in A(D)$, we have that $\beta(f_{(v_{\text{out}},u)}) \leq \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \leq 1 \leq c_{(v_{\text{out}},u)}$. Lastly, we have that $\beta(f_{(t,s)}) = \sum_{P \in \mathcal{P}(A,B)} \alpha(z_P) \leq \sum_{v \in V(G)} \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \leq |V(G)| = c_{(t,s)}$.

Finally, we verify that $\texttt{cost}(\beta) = -\texttt{cost}(\alpha)$. For this purpose, observe that

$$\begin{aligned}
\texttt{cost}(\beta) &= \sum_{(u,v) \in A(D)} k_{(u,v)} \beta(f_{(u,v)}) \\
&= \sum_{(v_{\text{in}},v_{\text{out}}) \in A(D)} k_{(v_{\text{in}},v_{\text{out}})} \beta(f_{(v_{\text{in}},v_{\text{out}})}) + k_{(t,s)} \beta(f_{(t,s)}) \\
&= \sum_{(v_{\text{in}},v_{\text{out}}) \in A(D)} x_v \beta(f_{(v_{\text{in}},v_{\text{out}})}) - \beta(f_{(t,s)}) \\
&= \sum_{v \in V(G)} x_v \sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) - \sum_{P \in \mathcal{P}(A,B)} \alpha(z_P) \\
&= -\sum_{P \in \mathcal{P}(A,B)} \alpha(z_P)(1 - \sum_{v \in V(P)} x_v) = -\texttt{cost}(\alpha).
\end{aligned}$$

This completes the proof. $\square$

To state the other direction of the relation, we first need the following folklore observation, whose correctness immediately follows from the constraints that encode the flow-preservation requirement.

Observation 5.2 (Folklore). *Given a feasible fractional solution β of* **Min-Cost Circulation LP**(D,c,k)*, there exists a function $\gamma_\beta : \mathcal{C} \to \mathbb{Q}_0^+$ where $\mathcal{C}$ is the set of all directed cycles in D, such that for every arc $(u,v) \in A(D)$, it holds that $\beta(f_{(u,v)}) = \sum_{\substack{C \in \mathcal{C} \\ \text{s.t. } (u,v) \in A(C)}} \gamma_\beta(C)$.*

Now, we state the other direction.

Lemma 5.7. *Let β be a feasible fractional solution of* **Min-Cost Circulation LP**(D,c,k) *where $(D,c,k) = \texttt{circulate}(G,x,A,B)$. For $\gamma = \gamma_\beta$ given by Observation 5.2, define $\alpha : \{z_P : P \in \mathcal{P}(A,B)\} \to \mathbb{Q}_0^+$ as follows.*

> *For every path $P \in \mathcal{P}(A,B)$, define $\alpha(z_P) = \gamma(C_P)$ where C_P is the directed cycle in D with arc set $\{(t,s)\} \cup \{(v_{\text{in}}, v_{\text{out}}) : v \in V(P)\} \cup \{(u_{\text{out}}, v_{\text{in}}) : (u,v) \in A(P)\}$.*

Then, α is a feasible fractional solution of **Penalized Flow Packing LP**(G,x,A,B) *whose cost is upper bounded by $-\texttt{cost}(\beta)$.*

Proof. To show that α is a feasible fractional solution of **Penalized Flow Packing LP**(G, x, A, B), we need to show that $\sum_{\substack{P\in\mathcal{P}(A,B)\\ \text{s.t. } v\in V(P)}} \alpha(z_P) \leq 1$ for all $v \in V(G)$. To this end, consider some vertex $v \in V(G)$. By the capacity constraints of **Min-Cost Circulation LP**, we have that $\beta(f_{(v_{\text{in}},v_{\text{out}})}) \leq c_{(v_{\text{in}},v_{\text{out}})} = 1$. However, by the choice of γ, we have that $\sum_{\substack{C\in\mathcal{C}\\ \text{s.t. } (v_{\text{in}},v_{\text{out}})\in A(C)}} \gamma(C) = \beta(f_{(v_{\text{in}},v_{\text{out}})})$, and hence $\sum_{\substack{C\in\mathcal{C}\\ \text{s.t. } (v_{\text{in}},v_{\text{out}})\in A(C)}} \gamma(C) \leq 1$. Here, $\mathcal{C}$ is the set of all directed cycles in D. By the definition of α, we have that $\sum_{\substack{P\in\mathcal{P}(A,B)\\ \text{s.t. } v\in V(P)}} \alpha(z_P) \leq \sum_{\substack{C\in\mathcal{C}\\ \text{s.t. } (v_{\text{in}},v_{\text{out}})\in A(C)}} \gamma(C)$. Therefore, $\sum_{\substack{P\in\mathcal{P}(A,B)\\ \text{s.t. } v\in V(P)}} \alpha(z_P) \leq 1$ as required.

Observe that the only arc of negative cost (given by k) in D is (t, s), and hence the only cycles of negative cost (in total) are the cycles C_P associated with paths $P \in \mathcal{P}(A, B)$. Thus, we have that

$$\begin{aligned}
\texttt{cost}(\alpha) &= \sum_{P\in\mathcal{P}(A,B)} \alpha(z_P)(1 - \sum_{v\in V(P)} x_v) \\
&= \sum_{P\in\mathcal{P}(A,B)} \gamma(C_P)(-k((t,s)) - \sum_{v\in V(P)} k((v_{\text{in}}, v_{\text{out}}))) \\
&= -\sum_{P\in\mathcal{P}(A,B)} \gamma(C_P)(\sum_{(u,v)\in A(C_P)} k((u,v)) \\
&\leq -\sum_{C\in\mathcal{C}} \gamma(C)(\sum_{(u,v)\in A(C)} k((u,v)) \\
&= -\sum_{(u,v)\in A(D)} k_{(u,v)}\beta(f_{(u,v)}) = -\texttt{cost}(\beta)
\end{aligned}$$

This completes the proof. □

It is well known that **Min-Cost Circulation LP** has at least one optimal solution that is integral, as stated in the following proposition.

Proposition 5.1 ([2]). **Min-Cost Circulation LP**(D, c, k) *admits an optimal (feasible) fractional solution that is integral.*

From Proposition 5.1, by Lemmas 5.6 and 5.7, we obtain the following corollary.

Corollary 5.1. Penalized Flow Packing LP(G, x, A, B) *admits an optimal (feasible) fractional solution that is integral.*

From Corollary 5.1, we obtain the following lemma.

Lemma 5.8. *Let* $\texttt{opt}$ *be the (fractional) optimum of* **Penalized Flow Packing LP**(G, x, A, B). *Then, G has* $\texttt{opt}$ *vertex-disjoint paths between A and B such that for each of these paths, say P, it holds that* $\sum_{v\in V(P)} x_v < 1$.

Proof. By Corollary 5.1, **Penalized Flow Packing LP**(G, x, A, B) admits an integral solution α of cost $\texttt{opt}$. If there exists a path $P \in \mathcal{P}(A, B)$ such that $\alpha(z_P) > 0$ and $\sum_{v\in V(G)} x_v \geq 1$, by modifying α to assign z_P the value 0 rather than $\alpha(z_P)$, we obtain another integral solution of the same (or better) cost. Thus, without loss of generality, assume that for every path $P \in \mathcal{P}(A, B)$ such that $\alpha(z_P) > 0$, it holds that $\sum_{v\in V(G)} x_v < 1$.

The constraints of the LP imply that for every path $P \in \mathcal{P}(A, B)$, either $\alpha(z_P) = 0$ or $\alpha(z_P) = 1$, and that all paths $P \in \mathcal{P}(A, B)$ such that $\alpha(z_P) = 1$ are pairwise vertex-disjoint. Let us denote $\mathcal{P}^\star = \{P \in \mathcal{P}(A, B) : \alpha(z_P) = 1\}$. Then, $\texttt{opt} = \sum_{P\in\mathcal{P}^\star}(1 - \sum_{v\in V(G)} x_v)$, which means that $|\mathcal{P}^\star| \geq \texttt{opt}$. □

Having Lemmas 5.5 and 5.8 at hand, the strong duality theorem leads us to the following result.

Lemma 5.9. *With respect to* **Pairwise-Flow Hitting LP**(G, w, h, t)*, let* $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$*. If* α *cannot be extended to a feasible fractional solution of* **Pairwise-Flow Hitting LP**(G, w, h, t)*, then there exists* $\overline{X} \in \texttt{ConPart}(G, h)$ *such that for all* $i, j \in \{1, 2, \ldots, h\}$ *with* $i < j$*, the graph* $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ *has* t *vertex-disjoint paths between* $\overline{X}[i]$ *and* $\overline{X}[j]$ *such that for each of these paths, say* P*, it holds that* $\sum_{v \in V(P)} \alpha(x_v) < 1$*.*

Proof. Suppose that there does not exist a feasible fractional solution of **Pairwise-Flow Hitting LP**(G, w, h, t) that extends α. By Lemma 5.5, there exists $\overline{X} \in \texttt{ConPart}(G, h)$ such that for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the optimum of **Penalized Flow Hitting LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), w, \alpha, \overline{X}[i], \overline{X}[j])$ is larger than t. By Observation 5.1, we infer that for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the optimum of **Penalized Flow Packing LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), w, \alpha, \overline{X}[i], \overline{X}[j])$ is larger than t as well. Thus, by Lemma 5.8, for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the graph $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ has t vertex-disjoint paths between $\overline{X}[i]$ and $\overline{X}[j]$ such that for each of these paths, say P, it holds that $\sum_{v \in V(P)} \alpha(x_v) < 1$. ☐

We are now ready to present the translation. Here, given a feasible fractional solution of **Grid Hitting LP**, the translation entails the multiplication of the value assigned to each variable x_v by $\mathcal{O}(t)$, along with the extension of the result to the variable set of **Pairwise-Flow Hitting LP**. We also pay the penalty of multiplying t by a fixed constant.

Lemma 5.10. *Let* H *be a graph with* $h = |V(H)|$*. There exist fixed constants* $c = c(H)$ *and* $d = d(H)$ *such that for any triple* (G, w, t) *where* G *is* H*-minor free, and any feasible fractional solution* α *of* **Grid Hitting LP**(G, w, t)*, the following holds. Define* α' *by* $\alpha'(x_v) = dt \cdot \alpha(x_v)$ *for all* $v \in V(G)$*. Then, there exists a feasible fractional solution* $\alpha^\star$ *of* **Pairwise-Flow Hitting LP**(G, w, h, ct) *that extends* α' *and such that* $\texttt{cost}(\alpha^\star) = dt \cdot \texttt{cost}(\alpha)$*.*

Proof. Let $c = c(H)$ and $d = d(H)$ be fixed constants determined later. Suppose, by way of contradiction, that there does not exist a feasible fractional solution of **Pairwise-Flow Hitting LP**(G, w, h, ct) that extends α'. Then, by Lemma 5.9, there exists $\overline{X} \in \texttt{ConPart}(G, h)$ such that for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the graph $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ has ct vertex-disjoint paths between $\overline{X}[i]$ and $\overline{X}[j]$ such that for each of these paths, say P, it holds that $\sum_{v \in V(P)} \alpha'(x_v) < 1$. For every choice of $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, let $\mathcal{P}_{i,j}$ denote the previously mentioned collection of ct paths. Moreover, let $\mathcal{P}^\star$ denote the collection of all paths in G that belong to $\mathcal{P}_{i,j}$ for some $i, j \in \{1, 2, \ldots, h\}$ with $i < j$.

Let us denote $U = \{v \in V(P) : P \in \mathcal{P}^\star\} \setminus X$. Then, the paths in $\mathcal{P}^\star$ exist in $G[U \cup X]$. By Lemma 5.4, this means that there exists a fixed constant $a = a(H)$ such that $\texttt{tw}(G[U]) > act$. From Proposition 4.3, by setting c to be c'/a where $c' = c'(H)$ is the constant in that proposition, $G[U]$ has a $t \times t$ grid as a minor.

Now, note that since $\sum_{v \in V(P)} \alpha'(x_v) < 1$ for all $P \in \mathcal{P}^\star$ and $|\mathcal{P}^\star| = \binom{h}{2}ct$, it holds that $\sum_{v \in U} \alpha'(x_v) < \binom{h}{2}ct$. By the definition of α, this means that $\sum_{v \in U} dt \cdot \alpha(x_v) < \binom{h}{2}ct$, and hence $\sum_{v \in U} \alpha(x_v) < \binom{h}{2}c/d$. By setting d to be $\binom{h}{2}c$, we have that $\sum_{v \in U} \alpha(x_v) < 1$. Because $G[U]$ has a $t \times t$ grid as a minor, this implies that there exists $S \in \texttt{Grid}_t(G)$ such that $\sum_{v \in S} \alpha(x_v) < 1$. However, this is a contradiction to the supposition that α if a feasible fractional solution of **Grid Hitting LP**(G, w, t).

So far, we derived that there exists a solution $\alpha^\star$ of **Pairwise-Flow Hitting LP**(G, w, h, ct) that extends α'. Observe that

$$\texttt{cost}(\alpha^\star) = \texttt{cost}(\alpha') = \sum_{v \in V(G)} w_v \alpha'(x_v) = dt \cdot \sum_{v \in V(G)} w_v \alpha(x_v) = dt \cdot \texttt{cost}(\alpha).$$

This completes the proof. ☐

6 From Pairwise-Flow Hitting LP to Well-Linkedness LP

In this section, we translate a feasible fractional solution of **Pairwise-Flow Hitting LP** to a feasible fractional solution of **Well-Linkedness LP**. For the sake of clarity, the translation is divided into two main phases.

6.1 Translation, Phase I: Fractional highly connected set

The objective of Phase I is to show that, for any infeasible fractional solution of **Well-Linkedness LP**, we can obtain a structured witness (based on Definition 4.1). The existence of this witness will be the foundation of Phase II, where the actual translation is made. We begin by modifying Definition 4.1 in two ways: first, we need to allow having, between every pair of vertices, many fractional paths rather than only one fractional path; second, we do not obtain a clique but only ensure that between any two large subsets of our structure, the flow is large. Formally, we introduce the following two definitions.

Definition 6.1. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. An (s,t,δ,κ)-*fractional well-linkedness witness *with respect to (G,x), or simply an (s,t,δ,κ)-*witness*, is a triple $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ where* (i) *$S \subseteq V(G)$ is of size s,* (ii) *H is a graph with $V(H) = S$,* (iii) *for each edge $e = \{u,v\} \in E(H)$, $\mathcal{Q}_e$ is a collection of paths in $\mathcal{P}_G(u,v)$, and* (iv) *$\lambda_e : \mathcal{Q}_e \to \mathbb{Q}^+$, such that the following conditions hold.*

1. *For every edge $e \in E(H)$ and $P \in \mathcal{Q}_e$, it holds that $\sum_{v \in V(P)} x_v \leq \delta$.*
2. *For every edge $e \in E(H)$, it holds that $\kappa \cdot (t/s^2) = \sum_{P \in \mathcal{Q}_e} \lambda_e(P)$.*
3. *For every vertex $v \in V(G)$, it holds that $\sum_{e \in E(H)} \sum_{\substack{P \in \mathcal{Q}_e \\ \text{s.t. } v \in V(P)}} \lambda_e(P) \leq 1$.*

Definition 6.2. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. An (s,t,δ,κ)-witness $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ is a* nice (s,t,δ,κ,η)-witness *(with respect to (G,x)) if the following condition holds: For every partition (T,T') of S, it holds that $|E_H(T,T')| \geq \eta \cdot |T| \cdot |T'|$.*

Similarly to Section 4.4, in the context of the definitions above, the term "with respect to (G,α)" where $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ is an abbreviation of "with respect to (G,x')" where $x'_v = \alpha(x_v)$ for every $v \in V(G)$. To make our goal clear, let us state one of the two main lemmas that we intend to prove in this section.

Lemma 6.1. *Let $\delta = 2$, $\kappa = \frac{(1-500\eta)^2}{10^7}$ and $0 < \eta < \frac{1}{500}$. For any triple (G,w,t) and $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ that cannot be extended to a feasible fractional solution of* **Well-Linkedness LP**(G,w,t)*, there exists a nice (s,t,δ,κ,η)-witness with respect to (G,α) for some $s \in \mathbb{N}$.*

Towards the proof of this lemma, let us first present our candidate for the subset S. Eventually, we will need to pick only a subset of it. To this end, we have the following corollary of Observations 4.1 and 4.2.

Corollary 6.1. *With respect to* **Well-Linkedness LP**(G,w,t)*, suppose that $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ cannot be extended to a feasible fractional solution. Then, there exists a subset $S \subseteq V(G)$ such that the (fractional) optimum of* **flow-LP**(G,S,α) *is larger than t.*

Now, we turn to analyze **flow-LP**(G,S,x). The following lemma reinterprets the meaning of a feasible fractional solution (of high value) to this LP, giving us an assignment that is easier to understand and analyze. Later, we work only with such simplified assignments. Here, it is convenient to think of the variables as follows: for all $U \subseteq S$, each vertex $u \in U$ sends $g_{U,v}$ units of flow to each vertex $v \in U$, where each variable $f_P^{(u,U),v}$ indicates how much of this flow is sent via the path $P \in \mathcal{P}_G(u,v)$. Roughly speaking, the conditions below will tell us that *(i)* a lot of flow is circulating, *(ii)* we only send it to "large" sets U, *(iii)* for all $U \subseteq S$, u sends exactly $g_{U,v}$ units of flow to v,[7] *(iv)* through every vertex, at most one unit of flow passes, and *(v)* we only send flow via "cheap" (in x) paths.

Lemma 6.2. *Let* **flow-LP**(G,S,x) *be of (fractional) optimum* $t^\star$. *Then, there exists a function* $\beta : \{g_{U,v} : U \subseteq S, v \in U\} \cup \{f_P^{(u,U),v} : U \subseteq S, u,v \in U, P \in \mathcal{P}_G(u,v)\} \to \mathbb{Q}_0^+$ *that has the following properties:*

1. $t^\star \leq \sum_{U\subseteq S}\sum_{u\in U}\beta(g_{U,u})(|U|-1)$.
2. *For all* $u \in S$ *and* $U \subseteq S$, *if* $\beta(g_{u,U}) > 0$ *then* $|U| > \frac{|S|}{2}$.
3. *For all* $U \subseteq S$ *and distinct* $u,v \in U$, *it holds that* $\sum_{P\in\mathcal{P}_G(u,v)}\beta(f_P^{(u,U),v}) = \beta(g_{U,u})$.
4. *For all* $r \in V(G)$, *it holds that* $\sum_{U\subseteq S}\sum_{\substack{u,v\in U\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}(\beta(f_P^{(u,U),v}) + \beta(f_P^{(v,U),u})) \leq 1$.
5. *For all* $U \subseteq S$, *distinct* $u,v \in U$ *and* $P \in \mathcal{P}_G(u,v)$, *if* $\beta(f_P^{(u,U),v}) > 0$ *then* $\sum_{r\in V(P)} x_r \leq 1$.

Proof. Let α be an optimal (feasible) fractional solution to **flow-LP**(G,S,x). Without loss of generality, we suppose that $\sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}}(\alpha(g_{U,u})+\alpha(g_{U,v})) = \sum_{P\in\mathcal{P}(u,v)}\alpha(f_P^{uv})$, for all $u,v \in S$. In addition, we suppose without loss of generality that if $\alpha(g_{u,U}) > 0$ then $|U| > \frac{|S|}{2}$, for all $U \subseteq S$ and $u \in U$. We define β as follows. For all $U \subseteq S$ and $u \in U$, set $\beta(g_{U,u}) = \alpha(g_{U,u})$. For all $U \subseteq S$, distinct $u,v \in U$ and $P \in \mathcal{P}_G(u,v)$, set

$$\beta(f_P^{(u,U),v}) = \frac{\alpha(g_{U,u})}{\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v}))}\cdot\alpha(f_P^{uv}).$$

Condition 2 immediately holds. To see that Condition 1 is satisfied, note that

$$t^\star = \texttt{cost}(\alpha) = \sum_{U\subseteq S}\sum_{u\in U}\alpha(g_{U,u})(|U|-\frac{|S|}{2}) \leq \sum_{U\subseteq S}\sum_{u\in U}\beta(g_{U,u})(|U|-1).$$

For Condition 2, recall that $\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v})) = \sum_{P\in\mathcal{P}(u,v)}\alpha(f_P^{uv})$, and therefore

$$\begin{aligned}\sum_{P\in\mathcal{P}_G(u,v)}\beta(f_P^{(u,U),v}) &= \sum_{P\in\mathcal{P}_G(u,v)}\left(\frac{\alpha(g_{U,u})}{\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v}))}\cdot\alpha(f_P^{uv})\right)\\ &= \frac{\alpha(g_{U,u})}{\sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U,u})+\alpha(g_{U,v}))}\cdot\sum_{P\in\mathcal{P}_G(u,v)}\alpha(f_P^{uv})\\ &= \alpha(g_{U,u}) = \beta(g_{U,u}).\end{aligned}$$

[7] In total, u can send more flow to v, but *in the context of* U, this is the amount of flow that it sends.

Next, for Condition 4, note that $\sum_{\substack{u,v\in S\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}(u,v)\\ \text{s.t. } r\in V(P)}}\alpha(f_P^{uv})\leq 1$ (since α is feasible). Therefore,

$$
\begin{aligned}
&\sum_{U\subseteq S}\sum_{\substack{u,v\in U\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\left(\beta(f_P^{(u,U),v})+\beta(f_P^{(v,U),u})\right)\\
&=\sum_{U\subseteq S}\sum_{\substack{u,v\in U\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\left(\frac{\alpha(g_{U,u})+\alpha(g_{U,v})}{\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v}))}\cdot\alpha(f_P^{uv})\right)\\
&=\sum_{\substack{u,v\in S\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}}\left(\frac{\alpha(g_{U,u})+\alpha(g_{U,v})}{\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v}))}\cdot\alpha(f_P^{uv})\right)\\
&=\sum_{\substack{u,v\in S\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\alpha(f_P^{uv})\cdot\left(\sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}}\frac{\alpha(g_{U,u})+\alpha(g_{U,v})}{\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v}))}\right)\\
&=\sum_{\substack{u,v\in S\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\alpha(f_P^{uv})\leq 1.
\end{aligned}
$$

Lastly, consider Condition 5 with respect to some $U\subseteq S$, distinct $u,v\in U$ and $P^\star\in\mathcal{P}_G(u,v)$ such that $\beta(f_{P^\star}^{(u,U),v})>0$. We need to show that $\sum_{r\in V(P^\star)}x_r\leq 1$. Since $\beta(f_{P^\star}^{(u,U),v})>0$, we have that $\alpha(g_{U,u})>0$ (by the definition of $\beta(f_{P^\star}^{(u,U),v})$). Then, let $\widehat{\alpha}$ be defined as α except that $\widehat{\alpha}(g_{U,u})=0$, $\widehat{\alpha}(g_{U\setminus\{v\},u})=\alpha(g_{U\setminus\{v\},u})+\alpha(g_{U,u})$ and for all $P\in\mathcal{P}_G(u,v)$,

$$
\widehat{\alpha}(f_P^{uv})=\alpha(f_P^{uv})-\frac{\alpha(f_P^{uv})}{\sum_{P'\in\mathcal{P}_G(u,v)}\alpha(f_{P'}^{uv})}\cdot\alpha(g_{U,u}).
$$

Let us verify that $\widehat{\alpha}$ is a feasible fraction solution to **flow-LP**(G,S,x). Since $\widehat{\alpha}(f_P^{uv})\leq\alpha(f_P^{uv})$ for all $P\in\mathcal{P}(u,v)$, and all other variables f_P^{ab} are assigned the same value by α and $\widehat{\alpha}$, the constraint $\sum_{a,b\in S}\sum_{\substack{P\in\mathcal{P}(a,b)\\ \text{s.t. } r\in V(P)}}f_P^{ab}\leq 1$ is satisfied by $\widehat{\alpha}$ for all $r\in V(G)$. Moreover, for all $a,b\in S$ such that $\{a,b\}\neq\{u,v\}$, it holds that $\sum_{\substack{U'\subseteq S\\ \text{s.t. } a,b\in U'}}(\widehat{\alpha}(g_{U',a})+\widehat{\alpha}(g_{U',b}))=\sum_{\substack{U'\subseteq S\\ \text{s.t. } a,b\in U'}}(\alpha(g_{U',a})+\alpha(g_{U',b}))$. Thus, the only constraint whose satisfaction by $\widehat{\alpha}$ is not immediate is as follows:

$$
\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(g_{U',u}+g_{U',v})\leq\sum_{P\in\mathcal{P}(u,v)}f_P^{uv}.
$$

Notice that

$$
\begin{aligned}
\sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\widehat{\alpha}(g_{U',u})+\widehat{\alpha}(g_{U',v})) &= \sum_{\substack{U'\subseteq S\\ \text{s.t. } u,v\in U'}}(\alpha(g_{U',u})+\alpha(g_{U',v}))-\alpha(g_{U,u})\\
&\leq \sum_{P\in\mathcal{P}(u,v)}\alpha(f_P^{uv})-\alpha(g_{U,u})\\
&= \sum_{P\in\mathcal{P}(u,v)}\widehat{\alpha}(f_P^{uv})+\sum_{P\in\mathcal{P}(u,v)}\frac{\alpha(f_P^{uv})}{\sum_{P'\in\mathcal{P}_G(u,v)}\alpha(f_{P'}^{uv})}\cdot\alpha(g_{U,u})-\alpha(g_{U,u})\\
&= \sum_{P\in\mathcal{P}(u,v)}\widehat{\alpha}(f_P^{uv}).
\end{aligned}
$$

Now, observe that $\mathtt{cost}(\widehat{\alpha})>\mathtt{cost}(\alpha)$, else we contradict the optimality of α. In addition, observe that

$$
\begin{aligned}
& \mathtt{cost}(\widehat{\alpha}) - \mathtt{cost}(\alpha) \\
= \; & (\widehat{\alpha}(g_{U\setminus\{v\},u}) - \alpha(g_{U\setminus\{v\},u}))(|U \setminus \{v\}| - \frac{|S|}{2}) + (\widehat{\alpha}(g_{U,u}) - \alpha(g_{U,u}))(|U| - \frac{|S|}{2}) \\
& - \sum_{P\in\mathcal{P}_G(u,v)} (\widehat{\alpha}(f_P^{uv}) - \alpha(f_P^{uv})) \left(\sum_{r\in V(P)} x_r\right) \\
= \; & -\alpha(g_{U,u}) + \sum_{P\in\mathcal{P}_G(u,v)} \left(\frac{\alpha(f_P^{uv})}{\sum_{P'\in\mathcal{P}_G(u,v)}\alpha(f_{P'}^{uv})} \cdot \alpha(g_{U,u})\right) \cdot \left(\sum_{r\in V(P)} x_r\right) \\
= \; & (\sum_{P\in\mathcal{P}_G(u,v)} \left(\sum_{r\in V(P)} x_r\right) - 1) \cdot \alpha(g_{U,u}).
\end{aligned}
$$

This means that if $\alpha(g_{U,u}) > 0$ then $\sum_{P\in\mathcal{P}_G(u,v)}\left(\sum_{r\in V(P)} x_r\right) \leq 1$. However, recall that indeed $\alpha(g_{U,u}) > 0$, and therefore $\sum_{P\in\mathcal{P}_G(u,v)}\left(\sum_{r\in V(P)} x_r\right) \leq 1$. This completes the proof. $\square$

We proceed to construct a "preliminary" witness. Roughly speaking, towards this goal, we change our view of sending flow from a single vertex u to a set of vertices U, and rather think of this scenario as if the vertices in U are all sending flow to each other via the vertex u. This view leads us to the definition of two functions as follows.

Definition 6.3. *Let* **flow-LP**(G, S, x) *be of (fractional) optimum $t^\star$. Let β be the function given by Lemma 6.2. Define $\gamma_\beta : \{\{u, v\} : u, v \in S, u \neq v\} \to \mathbb{Q}_0^+$ as follows. For all distinct $u, v \in S$,*

$$\gamma_\beta(\{u, v\}) = \sum_{\substack{U\subseteq S \\ \text{s.t. } u,v\in U}} \sum_{w\in U\setminus\{u,v\}} \frac{\beta(g_{U,w})}{|U| - 2}.$$

In order to define the second function, let us first state the following lemma whose correctness will follow from Condition 3 in Lemma 6.2. Here, for all distinct $u, v \in S$, $P \in \mathcal{P}_G(u, v)$ (or $P \in \mathcal{W}_G(u, v)$) and $w \in U \setminus \{u, v\}$ such that $w \in V(P)$, we let $P[u, w]$ and $P[w, v]$ denote the subpaths (or subwalks) of P between u and w, and between w and v, respectively.

Lemma 6.3. *Let* **flow-LP**(G, S, x) *be of (fractional) optimum $t^\star$. Let β be the function given by Lemma 6.2. For all $U \subseteq S$ and $w \in U$, there exists a function $\tau_\beta^{w,U} : \mathcal{W}_G(U\setminus\{w\}, U\setminus\{w\}) \to \mathbb{Q}_0^+$ such that the following conditions hold.*

1. *For all $P \in \mathcal{W}_G(U \setminus \{w\}, U \setminus \{w\})$ such that P does not consist of two paths both having w as an endpoint and whose other endpoints are distinct:*[8] *$\tau_\beta^{w,U}(P) = 0$.*

2. *For all $u \in U \setminus \{w\}$ and $P \in \mathcal{P}_G(w, u)$:*

$$\beta(f_P^{(w,U),u}) = \sum_{v\in U\setminus\{w,u\}} \sum_{\substack{P'\in\mathcal{W}_G(u,v) \\ \text{s.t. } w\in V(P'),P=P'[u,w]}} \tau_\beta^{w,U}(P').$$

3. *For all distinct $u, v \in U \setminus \{w\}$:* $\displaystyle\sum_{P\in\mathcal{W}_G(u,v)} \tau_\beta^{w,U}(P) = \beta(g_{U,w})/(|U| - 2).$

[8]That is, w is visited by P exactly once, the endpoints of P are distinct, and if we traverse P from any endpoint to w, we obtain a (simple) path.

Proof. Let $U \subseteq S$ and $w \in U$. We define $\tau_\beta^{w,U} : \mathcal{W}_G(U \setminus \{w\}, U \setminus \{w\}) \to \mathbb{Q}_0^+$ as follows. First, for all $P \in \mathcal{W}_G(U \setminus \{w\}, U \setminus \{w\})$ such that P does not consist of two paths both having w as an endpoint and whose other endpoints are distinct, define $\tau_\beta^{w,U}(P) = 0$. Thus, it is clear that the first property is satisfied.

Next, consider two distinct vertices $u, v \in U \setminus \{w\}$. From Condition 3 in Lemma 6.2, we know that

$$\sum_{P \in \mathcal{P}_G(w,u)} \beta(f_P^{(w,U),u}) = \sum_{P \in \mathcal{P}_G(w,v)} \beta(f_P^{(w,U),v}) = \beta(g_{U,w}).$$

The first equality in this condition implies that there exists a function $\widehat{\tau}_{u,v} : \mathcal{P}_G(w,u) \times \mathcal{P}_G(w,v) \to \mathbb{Q}_0^+$ such that

- for all $P \in \mathcal{P}_G(w,u)$, $\displaystyle\sum_{P' \in \mathcal{P}_G(w,v)} \widehat{\tau}_{u,v}((P,P')) = \beta(f_P^{(w,U),u})/(|U|-2)$, and
- for all $P \in \mathcal{P}_G(w,v)$, $\displaystyle\sum_{P' \in \mathcal{P}_G(w,u)} \widehat{\tau}_{u,v}((P',P)) = \beta(f_P^{(w,U),v})/(|U|-2)$.

Thus, for each $P \in \mathcal{W}_G(u,v)$ that consist of two paths both having w as an endpoint, we define $\tau_\beta^{w,U}(P) = \widehat{\tau}_{u,v}(P[w,u], P[w,v])$.

Observe that for all $u \in U \setminus \{w\}$ and $P \in \mathcal{P}_G(w,u)$, we have that

$$\begin{aligned}
\sum_{v \in U \setminus \{w,u\}} \sum_{\substack{P' \in \mathcal{W}_G(u,v) \\ \text{s.t. } w \in V(P'), P = P'[u,w]}} \tau_\beta^{w,U}(P') &= \sum_{v \in U \setminus \{w,u\}} \sum_{\substack{P' \in \mathcal{W}_G(u,v) \\ \text{s.t. } w \in V(P'), \\ P = P'[u,w], \\ P'[w,v] \in \mathcal{P}_G(w,v)}} \widehat{\tau}_{u,v}(P, P'[w,v]) \\
&= \sum_{v \in U \setminus \{w,u\}} \sum_{P' \in \mathcal{P}_G(w,v)} \widehat{\tau}_{u,v}(P,P') \\
&= \sum_{v \in U \setminus \{w,u\}} \beta(f_P^{(w,U),u})/(|U|-2) = \beta(f_P^{(w,U),u}).
\end{aligned}$$

Thus, the second property is satisfied. For the third property, note that for all distinct $u, v \in U \setminus \{w\}$, we have that

$$\begin{aligned}
\sum_{P \in \mathcal{W}_G(u,v)} \tau_\beta^{w,U}(P) &= \sum_{\substack{P \in \mathcal{W}_G(u,v) \\ \text{s.t. } w \in V(P), \\ P[w,u] \in \mathcal{P}(w,u), \\ P[w,v] \in \mathcal{P}(w,v)}} \widehat{\tau}_{u,v}(P[w,u], P[w,v]) \\
&= \sum_{P \in \mathcal{P}_G(w,u)} \sum_{P' \in \mathcal{P}_G(u,v)} \widehat{\tau}_{u,v}(P,P') \\
&= \sum_{P \in \mathcal{P}_G(w,u)} \beta(f_P^{(w,U),u})/(|U|-2) \\
&= \beta(g_{U,w})/(|U|-2).
\end{aligned}$$

This completes the proof. $\square$

In the context of Lemma 6.3, given $U \subseteq S$ and $w \in U$ with $\beta(g_{U,w}) > 0$, we let $\tau_\beta^{w,U}$ be an arbitrarily chosen function that has the property in this observation. Now, we can present the definition of the second function.

Definition 6.4. *Let* **flow-LP**(G,S,x) *be of (fractional) optimum* $t^\star$*. Let* β *be the function given by Lemma 6.2. Define* $\widehat{\rho}_\beta : \mathcal{W}_G(S,S) \to \mathbb{Q}_0^+$ *as follows. For all distinct* $u, v \in S$ *and* $P \in \mathcal{W}_G(u,v)$*,*

$$\widehat{\rho}_\beta(P) = \sum_{\substack{U \subseteq S \\ \text{s.t. } u,v \in U}} \sum_{\substack{w \in U \setminus \{u,v\} \\ \text{s.t. } w \in V(P)}} \tau_\beta^{w,U}(P).$$

Let us now prove that these functions satisfy useful properties.

Lemma 6.4. *Let* **flow-LP**(G, S, x) *be of (fractional) optimum* $t^\star$. *Let* β *be the function given by Lemma 6.2. Then, the following conditions are satisfied.*

1. $t^\star/2 \leq \sum_{\substack{u,v\in S\\ u\neq v}} \gamma_\beta(\{u,v\})$.

2. *For all (distinct)* $u, v \in S$, *it holds that* $\gamma_\beta(\{u,v\}) \leq \dfrac{100}{|S|^2} \displaystyle\sum_{\substack{u',v'\in S\\ u'\neq v'}} \gamma_\beta(\{u',v'\})$.

3. *For all (distinct)* $u, v \in S$, *it holds that* $\sum_{P\in\mathcal{W}_G(u,v)} \widehat{\rho}_\beta(P) = \gamma_\beta(\{u,v\})$.

4. *For all* $r \in V(G)$, *it holds that* $\sum_{\substack{u,v\in S\\ u\neq v}} \sum_{\substack{P\in\mathcal{W}_G(u,v)\\ \text{s.t. } r\in V(P)}} \widehat{\rho}_\beta(P) \leq 1$.

5. *For all (distinct)* $u, v \in S$ *and* $P \in \mathcal{W}_G(u,v)$, *if* $\widehat{\rho}_\beta(P) > 0$ *then* $\sum_{r\in V(P)} x_r \leq 2$.

Proof. To prove the inequalities and equalities that follow, we make use of conditions stated in Lemmas 6.2 and 6.3, and of Definitions 6.3 and 6.4. First, note that

$$\begin{aligned}
\sum_{U\subseteq S}\sum_{w\in U}\beta(g_{U,w})(|U|-1) &= \sum_{U\subseteq S}\sum_{w\in U}\sum_{\substack{u,v\in U\setminus\{w\}\\ u\neq v}} \frac{\beta(g_{U,w})(|U|-1)}{\binom{|U|-1}{2}}\\
&= 2\sum_{\substack{u,v\in S\\ u\neq v}}\sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}}\sum_{w\in U\setminus\{u,v\}}\frac{\beta(g_{U,w})}{|U|-2}\\
&= 2\sum_{\substack{u,v\in S\\ u\neq v}}\gamma_\beta(\{u,v\}). && \text{(Definition 6.3)}
\end{aligned}$$

Let us refer to the inequality above as Inequality (*).

Condition 1. Now, for Condition 1, note that

$$\begin{aligned}
t^\star &\leq \sum_{U\subseteq S}\sum_{w\in U}\beta(g_{U,w})(|U|-1) && \text{(Condition 1 in Lemma 6.2)}\\
&= 2\sum_{\substack{u,v\in S\\ u\neq v}}\gamma_\beta(\{u,v\}). && \text{(Inequality (*))}
\end{aligned}$$

Condition 2. For Condition 2, for all (distinct) $u, v \in S$, note that

$$\begin{aligned}
\gamma_\beta(\{u,v\}) &= \sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}}\sum_{w\in U\setminus\{u,v\}}\frac{\beta(g_{U,w})}{|U|-2} && \text{(Definition 6.3)}\\
&\leq \sum_{\substack{U\subseteq S\\ \text{s.t. } |U|\geq 3}}\sum_{w\in U}\frac{\beta(g_{U,w})}{|U|-2}.
\end{aligned}$$

In what follows, we suppose that $|S| \geq 3$, else $\gamma_\beta(\{u,v\}) = 0$ and the proof of the condition is complete. By Condition 2 in Lemma 6.2, if $\beta(g_{U,w}) > 0$ then $|U| > |S|/2$. Thus,

$$
\begin{aligned}
\gamma_\beta(\{u,v\}) &\leq \sum_{\substack{U\subseteq S\\ \text{s.t.} |U|>|S|/2}} \sum_{w\in U} \frac{\beta(g_{U,w})}{\max(\frac{|S|}{2},3)-2} \\
&= \frac{2}{\max(|S|,6)-4} \cdot \sum_{\substack{U\subseteq S\\ \text{s.t.} |U|>|S|/2}} \sum_{w\in U} \beta(g_{U,w}) \\
&\leq \frac{25(|S|-2)}{|S|^2} \cdot \sum_{\substack{U\subseteq S\\ \text{s.t.} |U|>|S|/2}} \sum_{w\in U} \beta(g_{U,w}) && \text{(Since } |S|\geq 3\text{)} \\
&= \frac{50}{|S|^2} \sum_{\substack{U\subseteq S\\ \text{s.t.} |U|>|S|/2}} \sum_{w\in U} \beta(g_{U,w})(\frac{|S|}{2}-1) \\
&\leq \frac{50}{|S|^2} \sum_{U\subseteq S} \sum_{w\in U} \beta(g_{U,w})(|U|-1) \\
&= \frac{100}{|S|^2} \sum_{\substack{u',v'\in S\\ u'\neq v'}} \gamma_\beta(\{u',v'\}). && \text{(Inequality (*))}
\end{aligned}
$$

Condition 3. For Condition 3, for all (distinct) $u, v \in S$, note that

$$
\begin{aligned}
\sum_{P\in\mathcal{W}_G(u,v)} \widehat{\rho}_\beta(P) &= \sum_{P\in\mathcal{W}_G(u,v)} \sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}} \sum_{\substack{w\in U\setminus\{u,v\}\\ \text{s.t. } w\in V(P)}} \tau_\beta^{w,U}(P) && \text{(Definition 6.4)} \\
&= \sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}} \sum_{w\in U\setminus\{u,v\}} \sum_{\substack{P\in\mathcal{W}_G(u,v)\\ \text{s.t. } w\in V(P)}} \tau_\beta^{w,U}(P) \\
&= \sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}} \sum_{w\in U\setminus\{u,v\}} \sum_{P\in\mathcal{W}_G(u,v)} \tau_\beta^{w,U}(P) && \text{(Condition 1 in Lemma 6.3)} \\
&= \sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}} \sum_{w\in U\setminus\{u,v\}} \frac{\beta(g_{U,w})}{|U|-2} && \text{(Condition 3 in Lemma 6.3)} \\
&= \gamma_\beta(\{u,v\}). && \text{(Definition 6.3)}
\end{aligned}
$$

Condition 4. For Condition 4, for all $r \in V(G)$, note that

$$
\begin{aligned}
&\sum_{\substack{u,v\in S\\ u\neq v}} \sum_{\substack{P\in\mathcal{W}_G(u,v)\\ \text{s.t. } r\in V(P)}} \widehat{\rho}_\beta(P) \\
&= \sum_{\substack{u,v\in S\\ u\neq v}} \sum_{\substack{P\in\mathcal{W}_G(u,v)\\ \text{s.t. } r\in V(P)}} \sum_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}} \sum_{\substack{w\in U\setminus\{u,v\}\\ \text{s.t. } w\in V(P)}} \tau_\beta^{w,U}(P) && \text{(Definition 6.4)} \\
&= \sum_{U\subseteq S} \sum_{\substack{u,v\in U\\ v\neq u}} \sum_{w\in U\setminus\{u,v\}} \sum_{\substack{P\in\mathcal{W}_G(u,v)\\ \text{s.t. } r,w\in V(P),\\ P[w,u]\in\mathcal{P}(w,u),\\ P[w,v]\in\mathcal{P}(w,v)}} \tau_\beta^{w,U}(P) && \text{(Condition 1 in Lemma 6.3)}
\end{aligned}
$$

Rearranging the above, we obtain that it is equal to

$$\sum_{U\subseteq S}\sum_{w\in U}\sum_{u\in U\setminus\{w\}}\sum_{\substack{P\in\mathcal{P}_G(w,u)\\ \text{s.t. } r\in V(P)}}\sum_{v\in U\setminus\{w,u\}}\Big(\sum_{P'\in\mathcal{P}_G(w,v)}\tau_\beta^{w,U}(P)-\sum_{\substack{P'\in\mathcal{P}_G(w,v)\\ \text{s.t. } r\in V(P')}}\tau_\beta^{w,U}(PP')\Big)$$
$$\leq\sum_{U\subseteq S}\sum_{w\in U}\sum_{u\in U\setminus\{w\}}\sum_{\substack{P\in\mathcal{P}_G(w,u)\\ \text{s.t. } r\in V(P)}}\sum_{v\in U\setminus\{w,u\}}\sum_{P'\in\mathcal{P}_G(w,v)}\tau_\beta^{w,U}(PP')$$
$$=\sum_{U\subseteq S}\sum_{\substack{w,u\in U\\ w\neq u}}\sum_{\substack{P\in\mathcal{P}_G(w,u)\\ \text{s.t. } r\in V(P)}}\sum_{v\in U\setminus\{w,u\}}\Big(\sum_{P'\in\mathcal{P}_G(w,v)}\tau_\beta^{w,U}(PP')+\sum_{P'\in\mathcal{P}_G(u,v)}\tau_\beta^{u,U}(PP')\Big)$$

Renaming w to u, u to v and v to w, we obtain that the above is equal to

$$\sum_{U\subseteq S}\sum_{\substack{u,v\in U\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\sum_{w\in U\setminus\{u,v\}}\Big(\sum_{P'\in\mathcal{P}_G(u,w)}\tau_\beta^{u,U}(PP')+\sum_{P'\in\mathcal{P}_G(v,w)}\tau_\beta^{v,U}(PP')\Big)$$
$$=\sum_{U\subseteq S}\sum_{\substack{u,v\in U\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}\sum_{w\in U\setminus\{u,v\}}\Big(\sum_{\substack{P'\in\mathcal{W}_G(v,w)\\ \text{s.t. } u\in V(P'),\\ P=P'[u,v]}}\tau_\beta^{u,U}(P')+\sum_{\substack{P'\in\mathcal{W}_G(u,w)\\ \text{s.t. } v\in V(P'),\\ P=P'[u,v]}}\tau_\beta^{v,U}(P')\Big).$$

By making use of Condition 2 in Lemma 6.3, and then of Condition 4 in Lemma 6.2, we obtain that the above is equal to

$$\sum_{U\subseteq S}\sum_{\substack{u,v\in U\\ u\neq v}}\sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}}(\beta(f_P^{(u,U),v})+\beta(f_P^{(v,U),u}))\leq 1.$$

Condition 5. Lastly, for Condition 5, choose (distinct) $u,v\in S$ and $P\in\mathcal{P}_G(u,v)$ such that $\widehat{\rho}_\beta(P)>0$. Note that $\widehat{\rho}_\beta(P)>0$ implies that $\sum\limits_{\substack{U\subseteq S\\ \text{s.t. } u,v\in U}}\sum\limits_{\substack{w\in U\setminus\{u,v\}\\ \text{s.t. } w\in V(P)}}\tau_\beta^{w,U}(P)>0$. Thus, there exist $U\subseteq S$ that includes u and v, and a vertex $w\in U\setminus\{u,v\}$ that belongs to $V(P)$, such that $\tau_\beta^{w,U}(P)>0$. Denote $\widehat{P}=P[u,w]$ and $\overline{P}=P[w,v]$. Now, recall that

$$\beta(f_{\widehat{P}}^{(w,U),u})=\sum_{v\in U\setminus\{w,u\}}\sum_{\substack{P'\in\mathcal{P}_G(u,v)\\ \text{s.t. } w\in V(P'),\widehat{P}=P'[u,w]}}\tau_\beta^{w,U}(P').$$

Since $\tau_\beta^{w,U}(P)$ occurs in the right side above, we derive that $\beta(f_{\widehat{P}}^{(w,U),u})>0$. Symmetrically, $\beta(f_{\overline{P}}^{(w,U),u})>0$. However, $\beta(f_{\widehat{P}}^{(w,U),u})>0$ implies that $\sum_{r\in V(\widehat{P})}x_r\leq 1$, and $\beta(f_{\overline{P}}^{(w,U),u})>0$ implies that $\sum_{r\in V(\overline{P})}x_r\leq 1$. Since $\sum_{r\in V(P)}x_r=\sum_{r\in V(\widehat{P})}x_r+\sum_{r\in V(\overline{P})}x_r-x_w$, we conclude that $\sum_{r\in V(P)}x_r\leq 2$. ☐

Since we would like to work with (simple) paths rather than walks, we modify Definition 6.4 as follows.

Definition 6.5. *Let* **flow-LP**(G,S,x) *be of (fractional) optimum* $t^\star$*. Let* β *be the function given by Lemma 6.2, and let* $\widehat{\rho}_\beta$ *be the function given by Definition 6.4. Let* $\rho'_\beta:\mathcal{W}_G(S,S)\rightarrow\mathcal{P}_G(S,S)$ *be an arbitrary function with the property that every* $P\in\mathcal{W}_G(S,S)$*, it holds that* $\rho'_\beta(P)$ *has the same endpoints as* P *and* $V(\rho'_\beta(P))\subseteq V(P)$.[9] *Define* $\rho_\beta:\mathcal{P}_G(S,S)\rightarrow\mathbb{Q}_0^+$ *as follows. For all distinct* $u,v\in S$ *and* $P\in\mathcal{P}_G(u,v)$*,*

$$\rho_\beta(P)=\sum_{P'\in{\rho'}_\beta^{-1}(P)}\widehat{\rho}_\beta(P').$$

[9]Given a path $P\in\mathcal{W}_G(S,S)$, we can attain such $\rho'_\beta(P)$ by computing a shortest path on the graph induced by the edges visited by P from one of its endpoint to the other.

Corollary 6.2. *Let* **flow-LP**(G, S, x) *be of (fractional) optimum* $t^\star$. *Let* β *be the function given by Lemma 6.2. Then, the following conditions are satisfied.*

1. $t^\star/2 \leq \sum_{\substack{u,v\in S\\u\neq v}} \gamma_\beta(\{u,v\})$.
2. *For all (distinct)* $u, v \in S$, *it holds that* $\gamma_\beta(\{u,v\}) \leq \dfrac{100}{|S|^2} \displaystyle\sum_{\substack{u',v'\in S\\u'\neq v'}} \gamma_\beta(\{u',v'\})$.
3. *For all (distinct)* $u, v \in S$, *it holds that* $\sum_{P\in\mathcal{W}_G(u,v)} \rho_\beta(P) = \gamma_\beta(\{u,v\})$.
4. *For all* $r \in V(G)$, *it holds that* $\sum_{\substack{u,v\in S\\u\neq v}} \sum_{\substack{P\in\mathcal{W}_G(u,v)\\ \text{s.t. } r\in V(P)}} \rho_\beta(P) \leq 1$.
5. *For all (distinct)* $u, v \in S$ *and* $P \in \mathcal{W}_G(u,v)$, *if* $\rho_\beta(P) > 0$ *then* $\sum_{r\in V(P)} x_r \leq 2$.

Proof. The first two conditions are stated in Lemma 6.4. Each of the other three conditions follows from the analogous condition in Lemma 6.4, based on Definition 6.5. $\square$

We normalize our functions γ and ρ to obtain the following corollary to Corollary 6.2.

Corollary 6.3. *Let* **flow-LP**(G, S, x) *be of (fractional) optimum larger than* t. *Then, there exist functions* $\gamma : \{\{u,v\} : u,v \in S, u \neq v\} \to \mathbb{Q}_0^+$ *and* $\rho : \mathcal{P}_G(S,S) \to \mathbb{Q}_0^+$ *that satisfy the following conditions.*

1. $t/2 = \sum_{\substack{u,v\in S\\u\neq v}} \gamma(\{u,v\})$.
2. *For all (distinct)* $u, v \in S$, *it holds that* $\sum_{P\in\mathcal{P}_G(u,v)} \rho(P) = \gamma(\{u,v\}) \leq 50t/|S|^2$.
3. *For all* $r \in V(G)$, *it holds that* $\sum_{\substack{u,v\in S\\u\neq v}} \sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}} \rho(P) \leq 1$.
4. *For all (distinct)* $u, v \in S$ *and* $P \in \mathcal{P}_G(u,v)$, *if* $\rho(P) > 0$ *then* $\sum_{r\in V(P)} x_r \leq 2$.

Proof. Let β be the function given by Lemma 6.2. Denote $c = \dfrac{2\sum_{\substack{u,v\in S\\u\neq v}} \gamma_\beta(\{u,v\})}{t}$. Note that $c \geq 1$. Define $\gamma : \{\{u,v\} : u,v \in S, u \neq v\} \to \mathbb{Q}_0^+$ as follows. For all (distinct) $u, v \in S$, define $\gamma(\{u,v\}) = \gamma(\{u,v\})/c$. Next, define $\rho : \mathcal{P}_G(S,S) \to \mathbb{Q}_0^+$ as follows. For all $P \in \mathcal{P}_G(S,S)$, define $\rho(P) = \rho_\beta(P)/c$. Then, because $c \geq 1$, the correctness of the corollary directly follows from Corollary 6.2. $\square$

We are now ready to construct the "preliminary" witness.

Lemma 6.5. *Let* **flow-LP**(G, S, x) *be of (fractional) optimum larger than* t. *Let* $\delta = 2$ *and* $\kappa = \frac{1}{10}$ *Then, there exists an* $(s, t, \delta, \kappa, \mu)$*-witness with respect to* (G, x) *where* $s = |S|$ *and* $|E(H)| \geq \mu \cdot s^2$ *for* $\mu = \frac{1}{1000}$.

Proof. Let γ and ρ be the functions whose existence is guaranteed by Corollary 6.3. We define a triple $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ as follows. First, note that $S \subseteq V(G)$ is already known. Now, we let H be the graph with $V(G) = S$ and $E(H) = \{\{u,v\} : u,v \in S, u \neq v, \kappa \cdot (t/s^2) \leq \gamma(\{u,v\})\}$. For each edge $e = \{u,v\} \in E(H)$, we define $\mathcal{Q}_e = \{P \in \mathcal{P}_G(u,v) : \rho(P) > 0\}$. Finally, for each edge $e = \{u,v\} \in E(H)$, we define $\lambda_e : \mathcal{Q}_e \to \mathbb{Q}^+$ as follows: For all $P \in \mathcal{Q}_e$, set $\lambda_e(P) = \rho(P)$.

To show that (S, H, Q) is an (s, t, δ, κ)-witness, we prove that the four conditions in Definition 6.1 are satisfied.

1. For every edge $e \in E(H)$ and $P \in \mathcal{Q}_e$, we need to show that $\sum_{r\in V(P)} x_r \leq \delta$. From Condition 4 in Corollary 6.3 it follows that $P \in \mathcal{Q}_e$ implies that $\rho(P) > 0$, and hence $\sum_{r\in V(P)} x_r \leq 2$.

2. For every edge $e = \{u, v\} \in E(H)$, we need to show that $\kappa\cdot(t/s^2) \leq \sum_{P\in\mathcal{Q}_e} \lambda_e(P)$. By the definition of $E(H)$, we have that $\kappa\cdot(t/s^2) \leq \gamma(\{u, v\})$. By Condition 2 in Corollary 6.3, we have that $\gamma(\{u, v\}) = \sum_{P\in\mathcal{P}_G(u,v)} \rho(P) = \sum_{P\in\mathcal{Q}_e} \lambda_e(P)$. In case $\kappa\cdot(t/s^2) < \sum_{P\in\mathcal{Q}_e} \lambda_e(P)$, we can clearly decrease the values assigned by λ_e (while keeping them being positive) until equality is achieved.

3. For every vertex $r \in V(G)$, we need to show that $\sum_{e\in E(H)} \sum_{\substack{P\in\mathcal{Q}_e\\ r\in V(P)}} \lambda_e(P) \leq 1$. By Condition 3 in Corollary 6.3, we have that $\sum_{e\in E(H)} \sum_{\substack{P\in\mathcal{Q}_e\\ r\in V(P)}} \lambda_e(P) \leq \sum_{\substack{u,v\in S\\ u\neq v}} \sum_{\substack{P\in\mathcal{P}_G(u,v)\\ \text{s.t. } r\in V(P)}} \rho(P) \leq 1$.

4. Lastly, we need to show that $|E(H)| \geq \mu \cdot s^2$. To this end, it is sufficient to prove that $d := |\{\{u, v\} : u, v \in S, u \neq v, \gamma(\{u, v\}) < \kappa \cdot (t/s^2)\}| < \binom{s}{2} - \mu \cdot s^2 = (\frac{1}{2} - \mu)s^2 - \frac{s}{2}$. By Condition 1 in Corollary 6.3, we have that $t/2 = \sum_{\substack{u,v\in S\\ u\neq v}} \gamma(\{u, v\})$. Moreover, by Condition 2 in Corollary 6.3, for all distinct $u, v \in S$, it holds that $\gamma(\{u, v\}) \leq 50 \cdot (t/s^2)$. Thus, it holds that $t/2 < d\cdot\kappa\cdot(t/s^2)+(\binom{s}{2}-d)\cdot 50(t/s^2)$, that is, $s^2 < 2(\kappa-50)d+100\binom{s}{2} < 2(\kappa - 50)d + 50s^2$. Thus, $d < 49s^2/(100 - 2\kappa) = 49s^2/(100 - \frac{1}{5}) < \frac{498}{1000}s^2 = (\frac{1}{2} - \mu)s^2 - s$ (for sufficiently large s).

This completes the proof. $\square$

To construct a nice witness, we need to modify our "preliminary" witness. To this end, we utilize the following lemma and its corollary, which will also be useful later in Section 6.2.

Lemma 6.6. *Let $0 < \eta < 1$ and $0 < \mu < 1$. Let G be a graph with $|E(G)| \geq \mu n^2$ (where $n = |V(G)|$). Then, there exists a set of edges $U \subseteq E(G)$ of size at most $\frac{\eta}{2}n^2$ and a connected component C of $G - U$ on vertex set S such that*

- *$|S| \geq (\mu - \frac{\eta}{2})n$.*
- *For any partition (T, T') of S, it holds that $|E_{G[S]}(T, T')| \geq \eta \cdot |T| \cdot |T'|$.*

Proof. The proof is based on the usage of a potential function. Let $\mathcal{G}$ contains all undirected graphs. Given a graph $H \in \mathcal{G}$, let $\mathcal{C}_H$ denote its set of connected components. We define a potential function $\varphi : \mathcal{G} \to \mathbb{N}_0$ as follows. Given a graph $H \in \mathcal{G}$, we define $\varphi(H) = \sum_{C\in\mathcal{C}_H} |V(C)|^2$. Initialize $U_1 := E(G)$ and $i := 1$. Now, as long as there exists a connected component $C_i \in \mathcal{C}_{G[U_i]}$ and a partition (T_i, T_i') of $V(C_i)$ such that $|E_{C_i}(T_i, T_i')| < \eta \cdot |T_i| \cdot |T_i'|$, we update $U_{i+1} := U_i \setminus E_{C_i}(T, T')$ and increment i by 1. Let $U^\star$ denote the set U_i obtained at the end of this process, that is, for the last index i.

In iteration i, the value of the potential function is $\varphi(G[U_i])$. Clearly, the process terminated before the value of the potential function becomes non-positive. In each iteration i, the value of the potential function decreases by $\varphi(G[U_i]) - \varphi(G[U_{i+1}]) = |V(C_i)|^2 - (|T_i|^2 + |T_i'|^2) = (|T_i| + |T_i'|)^2 - (|T_i|^2 + |T_i'|^2) = 2|T_i||T_i'|$, while the number of edges decreases by at most $|U_i| - |U_{i+1}| < \eta|T_i||T_i'|$. In particular, since the initial value of the potential is at most n^2, we have that $\sum_i 2|T_i||T_i'| \leq n^2$ (where i ranges over all indices considered during the process). Therefore, the total number of edges removed is smaller than $\sum_i \eta|T_i||T_i'| \leq \frac{\eta}{2}n^2$. This means that the number of edges we are left with is $|U^\star| > |E(G)| - \frac{\eta}{2}n^2 \geq (\mu - \frac{\eta}{2})n^2$.

Let C be the largest connected component in $\mathcal{C}_{G[U^\star]}$, and define $S = V(C)$. Clearly, by the condition of termination of our process, for any partition (T, T') of S, it holds that

$|E_{G[S]}(T,T')| > \eta \cdot |T| \cdot |T'|$. The average degree of a vertex in $G[U^\star]$ is $\dfrac{|U^\star|}{n} > \dfrac{(\mu - \frac{\eta}{2})n^2}{n} = (\mu - \dfrac{\eta}{2})n$. Therefore, there exists a vertex in $G[U^\star]$ of degree larger than $(\mu - \frac{\eta}{2})n$, which means that $|S| > (\mu - \frac{\eta}{2})n$. $\square$

Corollary 6.4. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be an (s, t, δ, κ)-witness with respect to (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$, such that $|E(H)| \geq \mu \cdot s^2$ for $\mu = \frac{1}{1000}$. Let $0 < \eta < 1$ satisfy $\mu - \frac{\eta}{2} > 0$. For some $s' \geq (\mu - \frac{\eta}{2})s$, $\delta' = \delta$ and $\kappa' = (\mu - \frac{\eta}{2})^2\kappa$, there exists $S' \subseteq S$ such that $(S', H' := H[S'], \mathsf{Q}' = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H')\})$ is a nice $(s', t, \delta', \kappa', \eta)$-witness with respect to (G, x).*

Proof. Apply Lemma 6.6 on H with μ and η, and let S' be the set that it outputs. Then, $|S'| = s' \geq (\mu - \frac{\eta}{2})s$, and the satisfaction of the condition in Definition 6.2 (by (S', H', Q')) follows immediately from Lemma 6.6. Moreover, since H' and Q' are restrictions of H and Q, respectively, it is immediate that Conditions 1 and 3 in Definition 6.1 are satisfied. Finally, note that for every edge $e \in E(H)$, it holds that $\kappa \cdot (t/s^2) = \sum_{P \in \mathcal{Q}_e} \lambda_e(P)$. Then, (S', H', Q') is a nice $(s', t, \delta, \widehat{\kappa}, \mu', \eta)$-witness for $\widehat{\kappa} = \kappa(s'/s)^2$. Observe that $\widehat{\kappa} = \kappa(s'/s)^2 \geq (\mu - \frac{\eta}{2})^2\kappa$ (since $s' \geq (\mu - \frac{\eta}{2})s$ and $\mu - \frac{\eta}{2} > 0$). Thus, by the definition of κ', Condition 2 is satisfied as well. (More precisely, it is satisfied with $=$ replaced by $\leq$. However, equality is easily ensured by decreasing values assigned by the functions λ_e.) $\square$

We are ready to conclude the proof of Lemma 6.1. For the sake of clarity, we restate it.

Lemma 6.1. *Let $\delta = 2$, $\kappa = \frac{(1-500\eta)^2}{10^7}$ and $0 < \eta < \frac{1}{500}$. For any triple (G, w, t) and $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ that cannot be extended to a feasible fractional solution of* **Well-Linkedness LP**(G, w, t)*, there exists a nice $(s, t, \delta, \kappa, \eta)$-witness with respect to (G, α) for some $s \in \mathbb{N}$.*

Proof. By Corollary 6.1, there exists a subset $S \subseteq V(G)$ such that the (fractional) optimum of **flow-LP**(G, S, α) is larger than t. By Lemma 6.5, there exists an $(s, t, 2, \frac{1}{10})$-witness with respect to (G, x), where $s = |S|$ and $|E(H)| \geq \mu \cdot s^2$ for $\mu = \frac{1}{1000}$. Thus, by Lemma 6.6 and since $(\frac{1}{1000} - \frac{\eta}{2})^2 \cdot \frac{1}{10} = \frac{(1-500\eta)^2}{10^7}$, there exists a nice $(s', t, 2, \frac{(1-50\eta)^2}{1000}, \frac{\eta}{4}, \eta)$-witness with respect to (G, x) for some $s' \geq (\frac{1}{1000} - \frac{\eta}{2})s$. $\square$

Before we turn to consider the second main result of this section, let us establish a lemma that concerns the result of applying Lemma 6.6 for special input graphs. Indeed, later we will be repeatedly modifying our graph, and would like it to still have the property (concerning the existence of many edges crossing any partition) in Lemma 6.6 while also ensuring a certain condition relating to deleted edges. Roughly speaking, in the following lemma we consider a graph $G^\star$ that already has the property described in Lemma 6.6, and delete a "small" set of edges, D, from it. Then, we identify a set S within the resulting graph G, so that $G[S]$ also has this property, and so that "many" edges from D have at least one endpoint in S.

Lemma 6.7. *Let $0 < \eta < \eta^\star < 1$, $0 < \mu < 1$ and $0 < \xi < 1$ such that $\sqrt{\frac{(\xi + \frac{\eta}{2})}{\eta^\star}} < \mu - \frac{\eta}{2}$ and $\sqrt{2\xi} - 2\xi > \eta$. Let $G^\star$ be a graph with $|E(G^\star)| \geq (\mu + \xi)n^2$, so that for any partition (T, T') of $V(G^\star)$, it holds that $|E_{G^\star}(T, T')| \geq \eta^\star \cdot |T| \cdot |T'|$. Let $D \subseteq E(G)$ have size at most ξn^2, and denote $G = G^\star - D$. Then, there exists a set of edges $U \subseteq E(G)$ of size at most $\frac{\eta}{2}n^2$ and a connected component C of $G - U$ on vertex set S such that the following two conditions hold.*

1. *Let $\widehat{\xi}n^2$ be the number of edges in D with exactly one endpoint in S. Then,*

$$|S| \geq (1 - \sqrt{\frac{(\widehat{\xi} + \frac{\eta}{2})}{\eta^\star}})n.$$

2. *For any partition (T, T') of S, it holds that $|E_{G[S]}(T, T')| \geq \eta \cdot |T| \cdot |T'|$.*

Proof. By Lemma 6.6, there exists a set of edges $U \subseteq E(G)$ of size at most $\frac{\eta}{2}n^2$ and a connected component C of $G - U$ on vertex set S such that

- $|S| \geq (\mu - \frac{\eta}{2})n$.
- For any partition (T, T') of S, it holds that $|E_{G[S]}(T, T')| \geq \eta \cdot |T| \cdot |T'|$.

Let $\widehat{\xi}n^2$ be the number of edges in D with exactly one endpoint in S. To complete the proof, let us show that $|S| \geq (1 - \sqrt{\frac{(\widehat{\xi} + \frac{\eta}{2})}{\eta^\star}})n$. Denote $|S| = \alpha n$ and $R = V(G^\star) \setminus S$ (note that $V(G^\star) = V(G)$). Then, $\eta^\star \cdot |S| \cdot |R| \leq |E_{G^\star}(S, R)|$, and hence $\eta^\star \cdot \alpha n \cdot (1-\alpha)n \leq |E_{G^\star}(S, R)|$. Since C is a connected component of $G - U$ and $|U| \leq \frac{\eta}{2}n^2$, we have that $|E_G(S, R)| \leq \frac{\eta}{2}n^2$, which means that $|E_{G^\star}(S, R)| \leq (\widehat{\xi} + \frac{\eta}{2})n^2$. Thus, $\alpha(1-\alpha) \leq \frac{(\widehat{\xi}+\frac{\eta}{2})}{\eta^\star}$. Let $\beta = \min\{\alpha, 1-\alpha\}$. Then, $\beta \leq \sqrt{\frac{(\widehat{\xi}+\frac{\eta}{2})}{\eta^\star}}$. However, $\alpha \geq \mu - \frac{\eta}{2}$ (because $|S| \geq (\mu - \frac{\eta}{2})n$). Since $\sqrt{\frac{(\widehat{\xi}+\frac{\eta}{2})}{\eta^\star}} \leq \sqrt{\frac{(\xi+\frac{\eta}{2})}{\eta^\star}} < \mu - \frac{\eta}{2}$, this means that $\beta = 1 - \alpha$, and therefore $\alpha \geq 1 - \sqrt{\frac{(\widehat{\xi}+\frac{\eta}{2})}{\eta^\star}}$ as required. ☐

Let us also phrase this lemma in terms of witnesses.

Corollary 6.5. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be a nice $(s, t, \delta, \kappa, \eta)$-witness with respect to (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$. Let $D \subseteq E(H)$ have size at most ξn^2. Let $0 < \eta' < \eta$ satisfy $\xi + \frac{\eta'}{2} + \sqrt{\frac{(\xi+\frac{\eta'}{2})}{\eta}} < |E(H)|/s^2$. Then, there exists $S' \subseteq S$ such that $(S', H' := (H - D)[S'], \mathsf{Q}' = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H')\})$ is a nice $(s', t, \delta, \kappa, \eta')$-witness with respect to $(G - D, x)$ such that $s' \geq (1 - \sqrt{\frac{(\widehat{\xi} + \frac{\eta'}{2})}{\eta}})s$ where $\widehat{\xi}s^2$ is the number of edges in D that have exactly one endpoint in S'.*

Proof. The corollary directly follows from Lemma 6.7 when applied on the graph H. ☐

In order to present the second main result of this section, we introduce the following definition. Here, we consider any large enough subsets of S (but which may not form a partition of S), and state that there is large flow between them that passes through "short paths", and so that no vertex is used "too much" by these paths.

Definition 6.6. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. A nice $(s, t, \delta, \kappa, \eta)$-witness $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ (with respect to (G, x)) is $(\gamma, \rho, \nu, \zeta)$-*useful *if for any two subsets $X, Y \subseteq S$ such that $|X|, |Y| \geq \gamma|S|$, there exists a set of paths $\mathcal{W}$ from $\mathcal{P}_H(X, Y)$, such that*

- *Any path in $\mathcal{W}$ contains at most ν edges.*
- *$|\mathcal{W}| \geq \frac{\gamma\rho}{2}n^2$.*
- *For any edge $e \in E(G)$, it holds that $\zeta \cdot |\{W \in \mathcal{W} : e \in E(W)\}| \leq 1$.*

Finally, we show that for the appropriate choice of parameters, a nice witness is in fact also a useful witness. This claim will follow as a corollary to the following general lemma.

Lemma 6.8. *Let $0 < \eta < 1$, $0 < \gamma < 1$. Denote $\rho = \dfrac{\gamma\eta}{2}$, $\zeta = (\dfrac{2}{\rho})^{\nu-1}$ and $\nu = \dfrac{4}{\eta\gamma}$. Let G be a graph that has the following property: for every partition (A, B) of $V(G)$, it holds that $|E_G(A, B)| \geq \eta \cdot |A| \cdot |B|$. Then, for any two subsets $X, Y \subseteq V(G)$ such that $|X|, |Y| \geq \gamma n$ (where $n = |V(G)|$), there exists a set of paths from $\mathcal{P}_G(X, Y)$ such that*

- *Any path in $\mathcal{W}$ contains at most ν edges.*
- $|\mathcal{W}| \geq \frac{\gamma\rho}{2} n^2$.
- *For any edge $e \in E(G)$, it holds that $|\{W \in \mathcal{W} : e \in E(W)\}| \leq \zeta$.*

Proof. Let us perform the following procedure. Initialize $i := 0$ and $X_0 := X$. Additionally, initialize $\mathcal{W}_1$ to be the *multiset* of paths that, for every vertex $v \in X$, contains ρn copies of the path on the single vertex v. Note that X_0 is a *set.* We will ensure that during the execution of the procedure, the following properties hold for any i corresponding to an earlier iteration:

1. $|X_i| \geq (\gamma + i \cdot \dfrac{\gamma\eta}{4})n$.
2. Every edge in $E(G[X_i])$ occurs in at most $(\dfrac{2}{\rho})^{i-1}$ paths in $\mathcal{W}_i$.
3. Each path in $\mathcal{W}_i$ contains at most i edges.
4. **Endpoints.** The paths in $\mathcal{W}_i$ satisfy the following properties.
   (a) Every path in $\mathcal{W}_i$ has a vertex in X as an endpoint.
   (b) For any vertex $x \in X$, there exist exactly ρn paths (including copies) in $\mathcal{W}_i$ whose only vertex is x.
   (c) For any vertex $x \in X_i \setminus X$, there exist at least ρn paths (including copies) in $\mathcal{W}_i$ such that x is an endpoint of that path.

Now, as long as there exist more than $|Y|/2$ vertices $y \in Y$ such that $y \notin X_i$, we execute the following steps:

1. Let B_i denote the bipartite graph with bipartition $(X_i, V(G) \setminus X_i)$ and whose edge set is $E_G(X_i, V(G) \setminus X_i)$.
2. Let N_i denote the set of vertices in $V(G)\setminus X_i$ whose degree in B_i is at least ρn. Accordingly, denote $B'_i = B_i[X_i \cup N_i]$.
3. For each vertex $x \in X_i$, let $\mathcal{P}_i^x$ denote the multiset that consists of $1/\rho$ copies of each occurrence of a path $P \in \mathcal{W}_i$ that has x as an endpoint and whose other endpoint (in case $|V(P)| \geq 2$) belongs to X.
4. Let $f_i : E(B'_i) \to \bigcup_{x \in X_i} \mathcal{P}_i^x$ be an arbitrarily chosen injective function such that for each edge $e = \{x, y\} \in E(B'_i)$ where $x \in X_i$ and $y \in N_i$, the path $f_i(e)$ belongs to $\mathcal{P}_i^x$. Such a function f_i exists because *(i)* every vertex $x \in X_i$ is incident to fewer than n vertices in B'_i (where $n = |V(G)|$), while *(ii)* $|\mathcal{P}_i^x| \geq n$ because $\mathcal{W}_i$ contains at least ρn occurrences of paths P that have x as an endpoint and whose other endpoint (in case $|V(P)| \geq 2$) belongs to X (under the assumption that invariant 4 is satisfied with respect to i), and each such occurrence gives rise to $1/\rho$ copies in $\mathcal{P}_i^x$.
5. For each edge $e \in E(B'_i)$, let $g_i(e)$ denote the path obtained by extending $f_i(e)$ with the edge e.
6. Update $X_{i+1} := X_i \cup N_i$, and $\mathcal{W}_{i+1} := \mathcal{W}_i \cup \{g_i(e) : e \in E(B'_i)\}$ where the number of occurrence of a path P in $\mathcal{W}_{i+1}$ is equal to its number of occurrences of P in $\mathcal{W}_i$ plus the number of edges $e \in E(B'_i)$ such that $g_i(e) = P$.
7. Increment i by 1.

Let us now show that the properties stated earlier are indeed preserved. This is proved by induction on i. The basis, where $i = 0$, is trivially true. In particular, invariant 1 holds since $|X| \geq \gamma n$. Now, suppose that the properties hold for i, and let us prove them for $i+1$.

By the inductive hypothesis, it is immediate that invariants 3 and 4 are satisfied. In particular, invariant 4 is satisfied because every vertex in N_i has at least ρn neighbors in B'_i and hence becomes the endpoint of at least ρn new paths added to $\mathcal{W}_{i+1}$. It is also easy to see that invariant 2 is satisfied. Indeed, since every edge in $E(G[X_i])$ occurs in at most $(\dfrac{2}{\rho})^{i-1}$ paths in $\mathcal{W}_i$, and every newly added path (to $\mathcal{W}_{i+1}$) consists of one out of $1/\rho$ copies of a path in $\mathcal{W}_i$ and a unique new edge from $E(B'_i)$, the following holds: For any edge $e \in E(G[X_{i+1}])$, the number of paths in $\mathcal{W}_{i+1}$ that contain e is either exactly 1 (if the edge belongs to $E(B'_i)$) or at most $(1 + \dfrac{1}{\rho})$ times the number of paths in $\mathcal{W}_i$ that contain it (if the edge belongs to $E(G[X_i])$). The latter number is upper bounded by $(1 + \dfrac{1}{\rho}) \cdot (\dfrac{2}{\rho})^{i-1} \leq (\dfrac{2}{\rho})^i$.

Now, we turn to prove invariant 1. By the inductive hypothesis, $|X_i| \geq (\gamma + i \cdot \frac{\gamma\eta}{4})n$. Thus, to obtain the bound $|X_{i+1}| \geq (\gamma + (i+1) \cdot \frac{\gamma\eta}{4})n$, it suffices to show that $|N_i| \geq \frac{\gamma\eta}{4} n$ (since $|X_{i+1} \setminus X_i| = |N_i|$). Since the process has not yet terminated, it holds that there are more than $|Y|/2$ vertices $y \in Y$ such that $y \notin X_i$. By Condition 4 and since $|Y| \geq \frac{\gamma}{2}n$, this means that $|V(G) \setminus X_i| \geq \frac{\gamma}{2}n$. Towards the proof that $|N_i| \geq \frac{\gamma\eta}{4}n$, let us denote $|X_i| = \beta n$. Then, $|V(G) \setminus X_i| = (1-\beta)n$. Due to the property of G stated in the lemma, we have that $|E(X_i, V(G) \setminus X_i)| \geq \eta\beta(1-\beta)n^2$. Thus, there exist at least $\frac{\eta}{2}(1-\beta)n$ vertices in $V(G) \setminus X_i$ such that each one among them has at least $\frac{\eta\beta}{2}n$ neighbors in X_i, since otherwise we obtain a contradiction to the inequality $|E(X_i, V(G) \setminus X_i)| \geq \eta\beta(1-\beta)n^2$ as follows: $|E(X_i, V(G) \setminus X_i)| < ((1-\frac{\eta}{2})(1-\beta)n) \cdot \frac{\eta\beta}{2}n + (\frac{\eta}{2}(1-\beta)n) \cdot \beta n = (\frac{1-\frac{\eta}{2}}{2} + \frac{1}{2}) \cdot (\eta\beta(1-\beta)n^2) = (1 - \frac{\eta}{4}) \cdot (\eta\beta(1-\beta)n^2) < \eta\beta(1-\beta)n^2$. Since $\beta \geq \gamma$ and $(1-\beta) \geq \frac{\gamma}{2}$, we get that there exists at least $\frac{\gamma\eta}{4}n$ vertices in $V(G) \setminus X_i$ such that each one among them has at least $\frac{\gamma\eta}{2}n$ neighbors in X_i. Because $\rho = \frac{\gamma\eta}{2}$, this precisely means that $|N_i| \geq \frac{\gamma\eta}{4}n$. This completes the proof that all conditions are preserved.

Notice that invariant 1 implies that the largest value i reaches is upper bounded by $\nu = \frac{4}{\eta\gamma}$. Let $X^\star$ and $\mathcal{W}^\star$ be the set of vertices and set of paths obtained at the last iteration. Then, since invariants 2, 3 and 4 are satisfied, we know that

1. Every edge in $E(G)$ occurs in at most $(\dfrac{2}{\rho})^{\nu-1}$ paths in $\mathcal{W}^\star$. In particular, this means that for any edge $e \in E(G)$, it holds that $|\{W \in \mathcal{W}^\star : e \in E(W)\}| \leq \zeta$.
2. Each path in $\mathcal{W}^\star$ contains at most ν edges.
3. **Endpoints.** The paths in $\mathcal{W}^\star$ satisfy the following properties.
   (a) Every path in $\mathcal{W}^\star$ has a vertex in X as an endpoint.
   (b) For any vertex $x \in X$, there exist exactly ρn paths (including copies) in $\mathcal{W}^\star$ whose only vertex is x.
   (c) For any vertex $x \in X^\star \setminus X$, there exist at least ρn paths (including copies) in $\mathcal{W}^\star$ such that x is an endpoint of that path.

We now proceed to define our candidate for the set $\mathcal{W}$ required to complete the proof of the lemma. Let $Y^\star$ be the set of vertices $y \in Y$ such that $y \in X^\star$. Then, $|Y^\star| \geq |Y|/2 \geq \frac{\gamma}{2}n$. For each vertex $y \in Y^\star$, let $\mathcal{P}_y$ be the set of paths in $\mathcal{W}^\star$ that have y as an endpoint. Note that for all $y \in Y^\star$, it holds that $|\mathcal{P}_y| \geq \rho n$. Now, we define $\mathcal{W} = \bigcup_{y \in Y^\star} \mathcal{P}_y$. Since $\mathcal{W} \subseteq \mathcal{W}^\star$, to conclude the proof, it remains to show that $|\mathcal{W}| \geq \frac{\gamma\rho}{2}n^2$. To this end, note that $|\mathcal{W}| \geq \sum_{y \in Y^\star} |\mathcal{P}_y| \geq |Y^\star| \cdot \rho n \geq \frac{\gamma\rho}{2}n^2$, and thus the proof is complete. $\square$

Corollary 6.6. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be a nice $(s, t, \delta, \kappa, \eta)$-witness with respect to (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$. Then, for any $0 < \gamma < 1$, it holds that (S, H, Q) is $(\gamma, \rho, \nu, \zeta)$-useful where $\rho = \frac{\gamma\eta}{2}$, $\zeta = (\frac{2}{\rho})^{\nu-1}$ and $\nu = \frac{4}{\eta\gamma}$.*

Proof. The corollary directly follows from Lemma 6.8 when applied on the graph H. □

6.2 Translation, Phase II: Iterative route detection

In this phase, we first describe a procedure whose starting point is Lemma 6.1 (the procedure also makes repeated calls to Corollary 6.4). During the execution of this procedure, we construct sets called $X_1, X_2, \ldots, X_h$ and $S_1, S_2, \ldots, S_h$ with special properties discussed later. Afterwards, we further utilize Corollary 6.6 to transfer flow between every pair of sets X_i and X_j via (subsets of) S_i and S_j. Then, we will turn to present the translation itself.

Towards the Construction of $\overline{\mathbf{X}}$ and $\overline{\mathbf{S}}$. The description of the promised procedure requires to establish several lemmas. To this end, we first need to define what is a one (or two) sided capacitated flow. We do not cap the flow passing through any vertex, but only the flow passing through each designated vertex *when its serves as the target endpoint of the flow-path* (the necessity of having sources and targets will be cleared when we prove Lemma 6.11 later). The reason why we need to cap flow will also be cleared later (when we reach Lemma 6.14). Here, the notation $\mathcal{P}_G^O(A, B)$ refers to the set of paths in G oriented from an endpoint $a \in A$ to an endpoint in $b \in B$. Note that if $a, b \in A \cap B$ ($a \neq b$), then a path oriented from a to b differs from the path on the same vertex set and edge set but with orientation from b to a.

Definition 6.7. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. Let $A, B \subseteq V(G)$. A pair $(\mathcal{F}, f)$, where $\mathcal{F}$ is collection of oriented paths in $\mathcal{P}_G^O(A, B)$ and $f : \mathcal{F} \to \mathbb{Q}^+$, is an* (A, B)-flow *if for every vertex $v \in V(G)$, it holds that $\sum_{\substack{F \in \mathcal{F} \\ \text{s.t. } v \in V(F)}} f(F) \leq 1$.*

- *The* power *of $(\mathcal{F}, f)$ is $\sum_{F \in \mathcal{F}} f(F)$.*
- *We say that $(\mathcal{F}, f)$ is* C-cheap *if for each path $F \in \mathcal{F}$, it holds that $\sum_{v \in V(F)} x_r \leq \mathsf{C}$.*
- *We say that $(\mathcal{F}, f)$ is K-capacitated if for each vertex $v \in B$, it holds that $\sum_{F \in \mathcal{F} \cap \mathcal{P}_G^O(A, v)} f(F) \leq \mathsf{K}$.*[10]

Now, we proceed to define what is a terminal set, and in particular what is an inclusion-wise minimal terminal set, with respect to Definition 6.7. This definition will be the basis of how our procedure will select the X_i's.

Definition 6.8. *Let G be a graph and $x : V(G) \to \mathbb{Q}_0^+$. Let $S \subseteq V(G)$. We say that a set $X \subseteq V(G)$ is a* $(\mathsf{P}, \mathsf{C}, \mathsf{K})$-terminal set *w.r.t. S if* (i) *$G[X]$ is a connected graph and* (ii) *there exists an (X, S)-flow $(\mathcal{F}, f)$ of power P that is C-cheap and K-capacitated.*

We say that X is inclusion-wise minimal *if there does not exist a strict subset $X' \subset X$ that is a $(\mathsf{P}, \mathsf{C}, \mathsf{K})$-terminal set w.r.t. S.*

Let us now show that an inclusion-wise minimal terminal set exists.

Lemma 6.9. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be a nice $(s, t, \delta, \kappa, \eta)$-witness with respect to (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$. Let $0 < \mathsf{P} \leq \mathsf{K} \cdot s$, $\mathsf{C} = \delta$ and $\mathsf{K} = \kappa \cdot \frac{t(s-1)}{s^2}$. Then, there exists an inclusion-wise minimal $(\mathsf{P}, \mathsf{C}, \mathsf{K})$-terminal set w.r.t. S.*

[10]Note that paths where v occurs as an internal vertex are not counted in this summation.

Proof. To show that there exists an inclusion-wise minimal $(\mathsf{P},\mathsf{C},\mathsf{K})$-terminal set w.r.t. S, it suffices to show that there exists an $(\mathsf{P},\mathsf{C},\mathsf{K})$-terminal set w.r.t. S, because minimality can be ensured by removing vertices one-by-one as long as we still have an $(\mathsf{P},\mathsf{C},\mathsf{K})$-terminal set w.r.t. S at hand.

Notice that because (S,H,Q) is nice, it holds that H is a connected graph. Since $\kappa>0$, for every edge $\{u,v\}\in E(H)$, there exists a path in G between u and v. Thus, we have that S is a subset of the vertex set of a single connected component, say C, of G. Then, let us denote $X=V(C)$. We claim that X is a $(\mathsf{P},\mathsf{C},\mathsf{K})$-terminal set w.r.t. S. Clearly, $G[X]$ is a connected graph. Now, we define an (X,S)-flow $(\mathcal{F},f)$ of power P that is C-cheap and K-capacitated as follows. For each vertex $v\in S$, we insert into $\mathcal{F}$ the path whose single vertex is v. Then, for each path $P\in\mathcal{F}$, we define $f(P)=\mathsf{K}$. With this definition, it is immediate that $(\mathcal{F},f)$ has power P and that it is K-capacitated. To show that it is C-cheap, it suffices to show that $x_v\leq\delta$ for any $v\in S$. However, since H is connected and for every edge $e\in E(H)$ and $P\in\mathcal{Q}_e$ (where $\mathcal{Q}_e\neq\emptyset$), it holds that $\sum_{v\in V(P)}x_v\leq\delta$, the claim follows. ☐

Next, we argue that because of the minimality of a terminal set, the removal of its vertices only "eliminates" little flow compared to the total amount of flow that realizes a fractional well-linkedness witness. To this end, we present a lemma that will help us to prove this claim.

Lemma 6.10. *Let G be a graph. Let $(\mathcal{F},f)$ be an (A,B)-flow for subsets $A,B\subseteq V(G)$. Let $X\subseteq V(G)$ such that $\sum_{\substack{F\in\mathcal{F}\\ \text{s.t. } X\cap V(F)\neq\emptyset}} f(F)\geq\mathsf{P}$. Then, there exists an (X,B)-flow $(\mathcal{F}',f')$ of power at least P and a surjective (w.r.t. $\mathcal{F}'$) function $g:\mathcal{F}\rightarrow\mathcal{F}'\cup\{\mathtt{nil}\}$ such that*

- *For any path $P\in\mathcal{F}$, it holds that $g(P')$ is either $\mathtt{nil}$ or a subpath of P oriented in the same direction and with the same target vertex.*
- *For any path $P'\in\mathcal{F}'$, it holds that $f'(P')=\sum_{P\in g^{-1}(P')}f(P)$.*

Moreover, if $(\mathcal{F},f)$ is C-cheap and K-capacitated, then so is $(\mathcal{F}',f')$.

Proof. For every vertex $v\in X$, let $\mathcal{F}_v$ denote the set of paths $P\in\mathcal{F}$ such that $v\in V(P)$. Now, for each vertex $v\in X$, select some subset $\mathcal{F}^\star_v\subseteq\mathcal{F}_v$, so that $\bigcup_{v\in X}\mathcal{F}^\star_v=\bigcup_{v\in X}\mathcal{F}_v$, and $\mathcal{F}^\star_v\cap\mathcal{F}^\star_u=\emptyset$ for all distinct $u,v\in X$. For every vertex $v\in X$, define $\widehat{\mathcal{F}}_v$ and a function $\widehat{f}_v:\widehat{\mathcal{F}}_v\rightarrow\mathbb{Q}^+$ as follows.

- We iterate over the paths $P^\star\in\mathcal{F}^\star_v$ in some (arbitrary) order. For each path $P^\star\in\mathcal{F}^\star_v$, execute the following steps:
  - Let $\widehat{P}$ be the subpath of $P^\star$ oriented from v to an endpoint, which is the target, of $P^\star$ (in the same direction in which $P^\star$ is oriented).
  - If $\widehat{P}$ has not been already inserted into $\widehat{\mathcal{F}}_v$, then we insert it and set $\widehat{f}_v(\widehat{P}):=f(P^\star)$.
  - Otherwise, update $\widehat{f}_v(\widehat{P}):=\widehat{f}_v(\widehat{P})+f(P^\star)$.

Now, we define $\mathcal{F}'=\bigcup_{v\in X}\widehat{\mathcal{F}}_v$. In addition, we define $f':\mathcal{F}'\rightarrow\mathbb{Q}^+$ as follows: For every $v\in X$ and $F\in\mathcal{F}^\star_v$, we define $f'(F)=\widehat{f}_v(F)$. Then, since $(\mathcal{F},f)$ is an (A,B)-flow, it is immediate that $(\mathcal{F}',f')$ is an (X,B)-flow as well. We define the function $g:\mathcal{F}\rightarrow\mathcal{F}'\cup\{\mathtt{nil}\}$ as follows. For every $P^\star\in\mathcal{F}$, if $V(P)\cap X=\emptyset$, then $g(P)=\mathtt{nil}$, and otherwise, let v be the vertex in X such that $P^\star\in\mathcal{F}^\star_v$, and let $g(P^\star)$ be the subpath $\widehat{P}$ of $P^\star$ defined in the iteration where $P^\star$ was considered. Note that the power of $(\mathcal{F}',f')$ equals $\sum_{\substack{F\in\mathcal{F}\\ \text{s.t. } X\cap V(F)\neq\emptyset}} f(F)$. Then, our construction immediately implies that g is surjective and satisfies the two properties in the lemma. In turn, the existence of this function g directly implies that if $(\mathcal{F},f)$ is C-cheap and K-capacitated, then so is $(\mathcal{F}',f')$: the cheapness of $(\mathcal{F}',f')$ follows from the cheapness of $(\mathcal{F},f)$ as well as the fact that g is surjective and the first property that it satisfies, while the capacitation of $(\mathcal{F}', f')$ follows from the capacitation of $(\mathcal{F}, f)$ as well as the fact that g is surjective and both of the properties that it satisfies. $\square$

We now proceed to prove that the removal of an inclusion-wise minimal terminal set indeed "eliminates" little flow.

Lemma 6.11. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be a nice $(s, t, \delta, \kappa, \eta)$-witness with respect to (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$. Let X be an inclusion-wise minimal $(\mathsf{P}, \mathsf{C}, \mathsf{K})$-terminal set w.r.t. S, where $0 < \mathsf{P} \leq \mathsf{K} \cdot s$, $\mathsf{C} = \delta$ and $\mathsf{K} = \kappa \cdot \frac{t(s-1)}{s^2}$. Then, it holds that $\sum_{e \in E(H)} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } X \cap V(Q) \neq \emptyset}} \lambda_e(Q) < \mathsf{P} + 1$.*

Proof. Suppose, by way of contradiction, that $\sum_{e \in E(H)} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } X \cap V(Q) \neq \emptyset}} \lambda_e(Q) \geq \mathsf{P} + 1$. Let $\mathcal{Q}$ be the set of paths $Q \in \bigcup_{e \in E(H)} \mathcal{Q}_e$ such that $X \cap V(Q) \neq \emptyset$. Orient each path in $\mathcal{Q}$ arbitrarily. For each path $Q \in \mathcal{Q}$, define $q(Q) = \lambda_e(Q)$ where $e \in E(H)$ is the edge whose endpoints are the endpoints of Q. Since for every vertex $v \in V(G)$, it holds that $\sum_{e \in E(H)} \sum_{\substack{P \in \mathcal{Q}_e \\ \text{s.t. } v \in V(P)}} \lambda_e(P) \leq 1$, we derive that $(\mathcal{Q}, q)$ is an (S, S)-flow. Since $\sum_{e \in E(H)} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } X \cap V(Q) \neq \emptyset}} \lambda_e(Q) = \sum_{Q \in \mathcal{Q}} q(Q)$, we derive that $(\mathcal{Q}, q)$ has power at least $\mathsf{P} + 1$. Since for every edge $e \in E(H)$ and $P \in \mathcal{Q}_e$, it holds that $\sum_{v \in V(P)} x_v \leq \delta$, we derive that $(\mathcal{Q}, q)$ is C-cheap. Lastly, since for every edge $e \in E(H)$, it holds that $\kappa \cdot (t/s^2) = \sum_{P \in \mathcal{Q}_e} \lambda_e(P)$, we also derive that $(\mathcal{Q}, q)$ is K-capacitated. By Lemma 6.10, this means that there exists an (X, S)-flow $(\mathcal{F}, f)$ of power $\mathsf{P} + 1$ that is C-cheap and K-capacitated.

Because $G[X]$ is a connected graph, we can arbitrarily choose a spanning tree T of $G[X]$. Let ℓ be some leaf of T. Now, denote $X' = X \setminus \{\ell\}$, $\mathcal{F}' = \mathcal{F} \cap \mathcal{P}_G^O(X', S)$ and $f' = f|_{\mathcal{F}'}$. Then, $G[X']$ is a connected graph, and it is clear that $(\mathcal{F}', f')$ is an (X', S)-flow that is C-cheap and K-capacitated. Since X is inclusion-wise minimal, this means that $(\mathcal{F}', f')$ has power smaller than P, that is, $\sum_{F \in \mathcal{F}'} f(F) < \mathsf{P}$. Moreover, since $(\mathcal{F}, f)$ is an (X, S)-flow, it holds that $\sum_{\substack{F \in \mathcal{F} \\ \text{s.t. } \ell \in \cap V(F)}} f(F) \leq 1$. Therefore, the power of $(\mathcal{F}, f)$, which equals $\sum_{F \in \mathcal{F}'} f(F) + \sum_{\substack{F \in \mathcal{F} \\ \text{s.t.} X \cap V(F) \neq \emptyset}} f(F)$, is smaller than $\mathsf{P} + 1$. Thus, we have reached a contradiction. $\square$

Since only little flow is eliminated in the settings of Lemma 6.11, we can maintain our witness so that is stays a fractional well-linkedness witness (with worse parameters) after the removal of an inclusion-wise minimal terminal set. Towards this, let us first show that only "few" edges lose "a lot" of flow when the terminal set is removed.

Lemma 6.12. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be a nice $(s, t, \delta, \kappa, \eta)$-witness w.r.t. (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$. Let X be an inclusion-wise minimal $(\mathsf{P}, \mathsf{C}, \mathsf{K})$-terminal set w.r.t. S, where $0 < \mathsf{P} \leq \mathsf{K} \cdot s$, $\mathsf{C} = \delta$ and $\mathsf{K} = \kappa \cdot \frac{t(s-1)}{s^2}$. Let $\Delta > 0$. Denote*

$$D = \{e \in E(H) : \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } X \cap V(Q) \neq \emptyset}} \lambda_e(Q) > \frac{\Delta(\mathsf{P} + 1)}{|E(H)|}\}.$$

Then, it holds that $|D| < |E(H)|/\Delta$.

Proof. By Lemma 6.11, it holds that $\sum_{e\in E(H)} \sum_{\substack{Q\in\mathcal{Q}_e \\ \text{s.t. } X\cap V(Q)\neq\emptyset}} \lambda_e(Q) < \mathsf{P}+1$. Therefore, we have that

$$\begin{aligned}
\mathsf{P}+1 &> \sum_{e\in E(H)} \sum_{\substack{Q\in\mathcal{Q}_e \\ \text{s.t. } X\cap V(Q)\neq\emptyset}} \lambda_e(Q) \\
&= \sum_{e\in D} \sum_{Q\in\mathcal{Q}_e} \lambda_e(Q) + \sum_{e\in E(H)\setminus D} \sum_{\substack{Q\in\mathcal{Q}_e \\ \text{s.t. } X\cap V(Q)\neq\emptyset}} \lambda_e(Q) \\
&\geq \sum_{e\in D} \sum_{Q\in\mathcal{Q}_e} \lambda_e(Q) \\
&> |D|\cdot\frac{\Delta(\mathsf{P}+1)}{|E(H)|}.
\end{aligned}$$

Thus, $|D| < |E(H)|/\Delta$. $\square$

Let us now show how to maintain our fractional well-linkedness witness. To this end, we have the following implication of Corollary 6.5. By Lemma 6.12, if we choose Δ to be large, then we lose only few edges, but the cap of the flow per edge decreases more.

Lemma 6.13. *Let $(S, H, \mathsf{Q} = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H)\})$ be a nice $(s, t, \delta, \kappa, \eta)$-witness w.r.t. (G, x) for some graph G and $x : V(G) \to \mathbb{Q}_0^+$. Let X be an inclusion-wise minimal $(\mathsf{P}, \mathsf{C}, \mathsf{K})$-terminal set w.r.t. S, where $0 < \mathsf{P} \leq \mathsf{K}\cdot s$, $\mathsf{C} = \delta$ and $\mathsf{K} = \kappa\cdot\frac{t(s-1)}{s^2}$. Let $\Delta > 0$ such that $\frac{\Delta(\mathsf{P}+1)}{|E(H)|} \leq \kappa(t/s)^2$. Denote*

$$D = \{e \in E(H) : \sum_{\substack{Q\in\mathcal{Q}_e \\ \text{s.t. } X\cap V(Q)\neq\emptyset}} \lambda_e(Q) > \frac{\Delta(\mathsf{P}+1)}{|E(H)|}\}.$$

Denote $\xi s^2 = |D|$. Let $0 < \eta' < \eta$ satisfy $\xi + \frac{\eta'}{2} + \sqrt{\frac{(\xi+\frac{\eta'}{2})}{\eta}} < |E(H)|/s^2$. Then, there exists $S' \subseteq S\setminus X$ such that $(S', H' := (H-D)[S'], \mathsf{Q}' = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H')\})$ is a nice $(s', t, \delta, \kappa', \eta')$-witness w.r.t. $(G - X, x|_{V(G)\setminus X})$ as follows:

- $s' \geq (1 - \sqrt{\frac{\widehat{\xi}+\eta'}{\eta}})s$ *where $\widehat{\xi}s^2$ is the number of edges in D with exactly one endpoint in S';*
- $\kappa' = (1 - \sqrt{\frac{\xi+\eta'}{\eta}})(\kappa - \frac{\Delta\mathsf{P}}{|E(H)|t/s^2})$.

Proof. By Corollary 6.5, there exists $S' \subseteq S$ such that $(S', H' := (H-D)[S'], \mathsf{Q}' = \{(\mathcal{Q}_e, \lambda_e) : e \in E(H')\})$ is a nice $(s', t, \delta, \kappa, \eta')$-witness with respect to $(G - D, x)$ such that $s' \geq (1 - \sqrt{\frac{(\widehat{\xi}+\frac{\eta'}{2})}{\eta}})s \geq (1 - \sqrt{\frac{(\widehat{\xi}+\eta')}{\eta}})s$ where $\widehat{\xi}s^2$ denotes the number of edges in D with exactly one endpoint in S'. Note that all of the edges in $E(H)$ that are incident to vertices in $X \cap S$ must belong to D. Thus, we can assume that $S' \cap X = \emptyset$. For every edge $e \in E(H')$, let us denote $\widehat{\mathcal{Q}}_e = \{Q \in \mathcal{Q}_e : X \cap V(Q) = \emptyset\}$, and $\widehat{\lambda}_e = \lambda_e|_{\widehat{\mathcal{Q}}_e}$. Let us denote $|E(H)| = \mu s^2$. Since none of the edges in D belongs to $E(H')$, it follows that for every edge $e \in E(H')$, it holds that

$$
\begin{aligned}
\sum_{P\in\widehat{\mathcal{Q}}_e} \lambda_e(P) &= \sum_{P\in\mathcal{Q}_e} \lambda_e(P) - \sum_{\substack{P\in\mathcal{Q}_e\\ \text{s.t. } X\cap V(P)\neq\emptyset}} \lambda_e(P)\\
&\geq \sum_{P\in\mathcal{Q}_e} \lambda_e(P) - \frac{\Delta(\mathsf{P}+1)}{|E(H)|}\\
&= \kappa\cdot(t/s^2) - \frac{\Delta(\mathsf{P}+1)}{\mu s^2}\\
&= \frac{s'^2}{s^2}(\kappa - \frac{\Delta(\mathsf{P}+1)}{\mu t})\cdot(t/s'^2)\\
&\geq (1-\sqrt{\frac{\xi+\eta'}{\eta}})^2(\kappa - \frac{\Delta(\mathsf{P}+1)}{\mu t})\cdot(t/s'^2)\\
&\geq (1-\sqrt{\frac{\xi+\eta'}{\eta}})(\kappa - \frac{\Delta\mathsf{P}}{\mu t})\cdot(t/s'^2).
\end{aligned}
$$

To achieve equality, it is clear that we can simply reduce values assigned by the functions $\widehat{\lambda}_e$. This implies $(S', H', \widehat{\mathsf{Q}} = \{(\widehat{\mathcal{Q}}_e, \widehat{\lambda}_e) : e \in E(H')\})$ is a nice $(s', t, \delta, \kappa', \eta')$-witness with respect to $(G - X, x|_{V(G)\setminus X})$ where $\kappa' = (1-\sqrt{\frac{\xi+\eta'}{\eta}})(\kappa - \frac{\Delta\mathsf{P}}{\mu t})$. ☐

The reason why we have not substituted $\widehat{\xi}$ by the upper bound ξ in the lemma above to lower bound s' is that this lower bound would not be good enough on its own. Specifically, if s' is as small as the size we get after substitution, we will need to resort to a lower bound on $\widehat{\xi}$ in order to say that we still have large flow from X to S'.

Procedure to Construct $\overline{\mathbf{X}}$ and $\overline{\mathbf{S}}$. We are now ready to describe the procedure, called `ConstructSetsAlg`, to "construct" $X_1, X_2, \ldots, X_h$ and $S_1, S_2, \ldots, S_h$. More precisely, we only show the existence of the desired sets. Here, we suppose that we have a triple (G, w, t) where G is an H-minor free graph, $h = |V(H)|$, and a function $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ that cannot be extended to a feasible fractional solution of **Well-Linkedness LP**(G, w, t). (The procedure will also construct a set S_{h+1}, which will be convenient to use in the analysis.)

1. By Lemma 6.1 with $\eta_1 = \frac{1}{10^3}$, there exists an $(s_1, t, \delta = 2, \kappa_1 = \frac{1}{4\cdot 10^7}, \eta_1)$-witness with respect to (G, x) for some $s_1 \in \mathbb{N}$. Let (S_1, H_1, Q_1) be such a witness.
2. Denote $G_1 := G$ and $x_1 := x$.
3. For $i = 1, 2, \ldots, h$:
   (a) Denote $\mu_i = |E(H_i)|/s_i^2$. Note that $\eta_i/4 \leq \mu_i \leq 1/2$ (the first inequality follows since (S_i, H_i, Q_i) is nice).
   (b) Denote $\mathsf{P}_i = p_i \cdot \kappa_i \cdot \eta_i \cdot \mu_i^2 \cdot t$ where $p_i = \frac{p_{i-1}^8 \eta_i^{12} \mu_i^{16}}{10^{20}}$. By Lemma 6.9, there exists an inclusion-wise minimal $(\mathsf{P}_i, \delta, \kappa_i \cdot \frac{(s_i-1)t}{s_i^2})$-terminal set w.r.t. S_i in G_i. Let X_i be such a witness.
   (c) Let $(\mathcal{F}_i, f_i)$ be an (X_i, S_i)-flow in G_i of power P_i that is δ-cheap and $\kappa_i \cdot \frac{(s_i-1)t}{s_i^2}$-capacitated.
   (d) Let $Y_i \subseteq S_i$ denote the set that contains every vertex that is the target of at least one path in $\mathcal{F}_i$.
   (e) Let $\Delta_i = 1/(2 \cdot p_i \cdot \eta_i \cdot \mu_i)$.

(f) Let $D_i = \{e \in E(H_i) : \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } X \cap V(Q) \neq \emptyset}} \lambda_e(Q) > \frac{\Delta_i(\mathsf{P}_i + 1)}{\mu_i s_i^2} = \frac{\kappa_i}{2} \cdot \frac{t}{s_i^2} + \frac{1}{2 p_i \eta_i \mu_i^2 s_i^2}\}$ with respect to Q_i.

(g) Denote $\xi_i = |D_i|/{s_i}^2$. By Lemma 6.12, $\xi_i \leq \frac{\mu_i}{\Delta_i} = 2 \cdot p_i \cdot \eta_i \cdot \mu_i^2$.

(h) Denote $G_{i+1} := G - X_i$ and $x_{i+1} := x_i|_{V(G_{i+1})}$.

(i) By Lemma 6.13, there exists a nice $(s_{i+1}, t, \delta, \kappa_{i+1}, \eta_{i+1})$-witness with respect to (G_{i+1}, x_{i+1}), say $(S_{i+1}, H_{i+1}, \mathsf{Q}_{i+1})$, such that

- $\eta_{i+1} := p_i^2 \cdot \eta_i^3 \cdot \mu_i^4/100$. Then, the condition $\xi_i + \frac{\eta_{i+1}}{2} + \sqrt{\frac{(\xi_i + \frac{\eta_{i+1}}{2})}{\eta_i}} < \mu_i$ of Lemma 6.13 is satisfied. Indeed, because $\mu_i \leq 1/2$, we have that $\xi_i + \frac{\eta_{i+1}}{2} + \sqrt{\frac{(\xi_i + \frac{\eta_{i+1}}{2})}{\eta_i}} \leq 2 p_i \eta_i \mu_i^2 + p_i^2 \eta_i^3 \mu_i^4/200 + \sqrt{2 p_i \mu_i^2 + p_i^2 \eta_i^2 \mu_i^4/200} < \mu_i$.
- $S_{i+1} \subseteq S_i \setminus X_i$,
- $s_{i+1} \geq (1 - \sqrt{\frac{\widehat{\xi}_i + \eta_{i+1}}{\eta_i}}) s_i = (1 - \sqrt{\frac{\widehat{\xi}_i}{\eta_i} + p_i^2 \eta_i^2 \mu_i^4/100}) s_i$ such that $\widehat{\xi}_i = |\widehat{D}_i|/s_i^2$ where $\widehat{D}_i$ is the set of edges in D_i with exactly one endpoint in S_{i+1}.
  Note that $s_{i+1} \geq (1 - \sqrt{\frac{\xi_i}{\eta_i} + p_i^2 \eta_i^2 \mu_i^4/100}) s_i \geq (1 - 2\mu_i \sqrt{p_i}) s_i$.
- $\kappa_{i+1} := (1 - \sqrt{\frac{\xi_i + \eta_{i+1}}{\eta_i}})^2 \kappa_i \geq (1 - 2\mu_i \sqrt{p_i})^2 \kappa_i \geq (1 - \sqrt{p_i})^2 \kappa_i \geq \frac{1}{2} \kappa_i$,

Finally, denote $X^\star = X_1 \cup X_2 \cup \cdots \cup X_h$, and for any $i \in \{1, 2, \ldots, h\}$, denote $X^{<i} = \bigcup_{j=1}^{i-1} X_j$, $X^{>i} = \bigcup_{j=i+1}^{h} X_j$, and $\overline{X}_i = X^\star \setminus X_i$. For convenience, denote $X^{<h+1} = X^\star$.

We will later require the following observation.

Observation 6.1. *For all $i \in \{1, 2, \ldots, h\}$, it holds that* $\frac{\kappa_{i+1} \cdot \frac{(s_{i+1}-1)t}{s_{i+1}^2}}{\kappa_i \cdot \frac{(s_i - 1)t}{s_i^2}} \leq 1$.

Proof. Consider some $i \in \{2, 3, \ldots, h+1\}$. Then, we have that

$$\begin{aligned}
\kappa_{i+1} \cdot \frac{(s_{i+1} - 1)t}{s_{i+1}^2} &= (1 - \sqrt{\frac{\xi_i + \eta_{i+1}}{\eta_i}})^2 \cdot \kappa_i \cdot \frac{(s_{i+1} - 1)t}{s_{i+1}^2} \\
&< (1 - \sqrt{\frac{\xi_i + \eta_{i+1}}{\eta_i}})^2 \cdot \kappa_i \cdot \frac{((1 - \sqrt{\frac{\xi_i + \eta_{i+1}}{\eta_i}}) s_i - 1)t}{(1 - \sqrt{\frac{\xi_i + \eta_{i+1}}{\eta_i}})^2 s_i^2} \\
&\leq \kappa_i \cdot \frac{(s_i - 1)t}{s_i^2}.
\end{aligned}$$

□

Flow Between Pairs of Sets in $\mathbf{\overline{X}}$. We proceed to further analyze the sets $X_1, X_2, \ldots, X_h$ and $S_1, S_2, \ldots, S_h$ that we have at hand. First, due to our capping, each set X_i does not only send large flow to S_i, but it sends this large flow to a linear fraction of S_i. However, we need a slightly stronger statement from this, namely, that each set X_i sends large flow to a linear fraction of S_i such that every vertex in this linear fraction is the target of "large" flow (rather than just nonzero flow). To this end, we have the following definition and corollary (that follows from the lemma given after the definition).

Definition 6.9. *Consider the settings of an execution of* `ConstructSetsAlg`*. For any* $i \in \{1, 2, \ldots, h\}$*, define*

$$\widetilde{Y}_i = \{y \in Y_i : \sum_{F \in \mathcal{F}_i \cap \mathcal{P}^O_{G_i}(X_i, y)} f_i(F) \geq \mathsf{P}_i/(2s_i)\}.$$

Lemma 6.14. *Let* G *be a graph. Let* $(\mathcal{F}, f)$ *be a* K*-capacitated* (A, B)*-flow of power at least* P *for* $A, B \subseteq V(G)$*. Denote* $\widetilde{B} = \{b \in B : \sum_{F \in \mathcal{F} \cap \mathcal{P}^O_G(A,b)} f(F) \geq \mathsf{P}_i/(2|B|)\}$*. Then,* $|\widetilde{B}| \geq \mathsf{P}/(2\mathsf{K})$*.*

Proof. Since $(\mathcal{F}, f)$ has power P, it holds that

$$\sum_{F \in \mathcal{F}} f(F) \geq \mathsf{P}.$$

Moreover, since $(\mathcal{F}, f)$ is K-capacitated, for each vertex $b \in B$, it holds that

$$\sum_{F \in \mathcal{F} \cap \mathcal{P}^O_G(A,b)} f(F) \leq \mathsf{K}.$$

However, note that

$$\begin{aligned} \textstyle\sum_{F \in \mathcal{F}} f(F) &= \sum_{b \in B \setminus \widetilde{B}} \sum_{F \in \mathcal{F} \cap \mathcal{P}^O_G(A,b)} f(F) + \sum_{b \in \widetilde{B}} \sum_{F \in \mathcal{F} \cap \mathcal{P}^O_G(A,b)} f(F) \\ &< |B \setminus \widetilde{B}| \cdot \mathsf{P}/(2|B|) + |\widetilde{B}| \cdot \mathsf{K} \\ &\leq \mathsf{P}/2 + |\widetilde{B}| \cdot \mathsf{K}. \end{aligned}$$

From the first and third inequalities, we derive that $\mathsf{P}/(2\mathsf{K}) \leq |\widetilde{B}|$. This completes the proof. $\square$

Corollary 6.7. *Consider the settings of an execution of* `ConstructSetsAlg`*. For any* $i \in \{1, 2, \ldots, h\}$*, it holds that* $|\widetilde{Y}_i| \geq \frac{p_i \eta_i \mu_i^2}{2} \cdot s_i$*.*

Proof. By Lemma 6.14, we have that

$$\begin{aligned} |\widetilde{Y}_i| &\geq \mathsf{P}_i/(2\kappa_i \cdot \frac{(s_i - 1)t}{s_i^2}) \\ &= (p_i \cdot \kappa_i \cdot \eta_i \cdot \mu_i^2 \cdot t)/(2\kappa_i \cdot \tfrac{(s_i-1)t}{s_i^2}) \\ &\geq \frac{p_i \eta_i \mu_i^2}{2} \cdot s_i. \end{aligned}$$

This completes the proof. $\square$

Now, we show that X_i does not only have large flow to S_i, but it even has large flow to S_{i+1}. Here we will make use of the tradeoff provided by the parameter $\widehat{\xi}_i$–namely, either $\widehat{\xi}_i$ itself is large or S_{i+1} is large.

Lemma 6.15. *Consider the settings of an execution of* `ConstructSetsAlg`*. For any* $i \in \{1, 2, \ldots, h\}$*, there exist a subset* $Z_i^\star \subseteq S_{i+1}$ *and an* $(X_i, Z_i^\star)$*-flow* $(\mathcal{F}_i^\star, f_i^\star)$ *in* G_i *with the three following properties.*

1. $|Z_i^\star| \geq \frac{p_i^2 \eta_i^3 \mu_i^4}{400} \cdot s_i$.
2. $(\mathcal{F}_i^\star, f_i^\star)$ *is* δ*-cheap and* $\kappa_i \cdot \frac{(s_i - 1)t}{s_i^2}$*-capacitated.*

3. *For every vertex $z \in Z_i^\star$, it holds that* $\displaystyle\sum_{F \in \mathcal{F}_i^\star \cap \mathcal{P}^O_{G_i}(X_i, z)} f_i^\star(F) \geq \frac{p_i^2 \kappa_i \eta_i^3 \mu_i^4}{400} \cdot \frac{t}{s_{i+1}}$.

Proof. By Corollary 6.7, we know that

$$|\widetilde{Y}_i| \geq \frac{p_i \eta_i \mu_i^2}{2} \cdot s_i.$$

Additionally, recall that

$$s_{i+1} \geq (1 - \sqrt{\frac{\widehat{\xi}_i}{\eta_i} + p_i^2 \eta_i^2 \mu_i^4 / 100}) s_i.$$

In light of these inequalities, we consider two cases as follows.

Case 1. First, suppose that $\widehat{\xi}_i \leq p_i^2 \eta_i^3 \mu_i^4 / 100$. In this case,

$$\begin{aligned} s_{i+1} &\geq (1 - \sqrt{\frac{p_i^2 \eta_i^3 \mu_i^4 / 100}{\eta_i} + p_i^2 \eta_i^2 \mu_i^4 / 100}) s_i \\ &\geq (1 - \frac{1}{9} p_i^2 \eta_i^2 \mu_i^4) s_i. \end{aligned}$$

Therefore, $|\widetilde{Y}_i \cap S_{i+1}| \geq \dfrac{1}{4} p_i \eta_i \mu_i^2 \cdot s_i$. Accordingly, we denote $Z_i^\star = \widetilde{Y}_i \cap S_{i+1}$, $\mathcal{F}_i^\star = \mathcal{F}_i \cap \mathcal{P}_{G_i}(X_i, Z_i^\star)$ and $f_i^\star = f_i|_{\mathcal{F}_i^\star}$. Because $(\mathcal{F}_i, f_i)$ is δ-cheap and $\kappa_i \cdot \dfrac{(s_i - 1)t}{s_i^2}$-capacitated, so is $(\mathcal{F}_i^\star, f_i^\star)$. Moreover, the satisfaction of Condition 3 in the lemma follows from the definition of $\widetilde{Y}_i$ because $\mathsf{P}_i / (2 s_i) \geq \dfrac{p_i^2 \kappa_i \eta_i^3 \mu_i^4}{400} \cdot \dfrac{t}{s_{i+1}}$.

Case 2. Second, suppose that $\widehat{\xi}_i > p_i^2 \eta_i^3 \mu_i^4 / 100$, that is, $|\widehat{D}_i| > \dfrac{p_i^2 \eta_i^3 \mu_i^4}{100} s_i^2$. In other words, there are more than $\dfrac{p_i^2 \eta_i^3 \mu_i^4}{100} s_i^2$ edges $e \in \widehat{D}_i \subseteq E(H_i)$ that have exactly one endpoint in S_{i+1} and such that

$$\sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } X_i \cap V(Q) \neq \emptyset}} \lambda_e(Q) > \frac{\kappa_i}{2} \cdot \frac{t}{s_i^2} + \frac{1}{2 p_i \eta_i \mu_i^2 s_i^2}.$$

Note that for $e \in E(H_i)$, by $\mathcal{Q}_e$ we refer to the set corresponding to the fractional well-linkedness witness associated with S_i. Let $\mathcal{Q}$ be the set of paths $Q \in \bigcup_{e \in \widehat{D}_i} \mathcal{Q}_e$. Orient each path in $\mathcal{Q}$ towards its endpoint in S_{i+1}. For each path $Q \in \mathcal{Q}$, define $q(Q) = \lambda_e(Q)$ where $e \in \widehat{D}_i$ is the edge whose endpoints are the endpoints of Q. Notice that $(\mathcal{Q}, q)$ is an (S_i, S_{i+1})-flow that is δ-cheap and $\kappa_i \cdot \frac{(s_i - 1)t}{s_i^2}$-capacitated. By Lemma 6.10, there exists an (X_i, S_{i+1})-flow $(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)$ of power larger than $\widehat{\mathsf{P}} = |\widehat{D}_i| \cdot (\frac{\kappa_i}{2} \cdot \frac{t}{s_i^2} + \frac{1}{2 p_i \eta_i \mu_i^2 s_i^2})$ that is δ-cheap and $\kappa_i \cdot \frac{(s_i - 1)t}{s_i^2}$-capacitated. Let us denote

$$Z_i^\star = \{v \in S_{i+1} : \sum_{F \in \widetilde{\mathcal{F}}_i \cap \mathcal{P}^O_{G_i}(X_i, v)} \widetilde{f}_i(F) \geq \widehat{\mathsf{P}} / (2 s_{i+1})\}.$$

By Lemma 6.14, we have that

$$\begin{aligned}
|Z_i^\star| &\geq \widehat{\mathsf{P}}/(2\kappa_i \cdot \frac{(s_i-1)t}{s_i^2}) \\
&= |\widehat{D}_i| \cdot (\frac{\kappa_i}{2} \cdot \frac{t}{s_i^2} + \frac{1}{2p_i\eta_i\mu_i^2 s_i^2})/(2\kappa_i \cdot \frac{(s_i-1)t}{s_i^2}) \\
&\geq |\widehat{D}_i| \cdot \frac{\kappa_i}{2} \cdot \frac{t}{s_i^2}/(2\kappa_i \cdot \frac{(s_i-1)t}{s_i^2}) \\
&\geq |\widehat{D}_i|/(4s_i) \\
&= \widehat{\xi}_i s_i/4 \\
&\geq \frac{p_i^2\eta_i^3\mu_i^4}{400} \cdot s_i.
\end{aligned}$$

Accordingly, we denote $\mathcal{F}_i^\star = \widetilde{\mathcal{F}}_i \cap \mathcal{P}_{G_i}(X_i, Z_i^\star)$ and $f_i^\star = \widetilde{f}_i|_{\mathcal{F}_i^\star}$. Because $(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)$ is δ-cheap and $\kappa_i \cdot \frac{(s_i-1)t}{s_i^2}$-capacitated, so is $(\mathcal{F}_i^\star, f_i^\star)$. To see that Condition 3 in the lemma it satisfied, note that for every $z \in Z_i^\star$, it holds that

$$\begin{aligned}
\sum_{F \in \widetilde{\mathcal{F}}_i \cap \mathcal{P}^O_{G_i}(X_i, v)} \widetilde{f}_i(F) &\geq \widehat{\mathsf{P}}/(2s_{i+1}) \\
&\geq |\widehat{D}_i| \cdot \tfrac{\kappa_i}{2} \cdot \tfrac{t}{s_i^2}/(2s_{i+1}) \\
&= \widehat{\xi}_i \kappa_i t/(4s_{i+1}) \\
&\geq \frac{p_i^2\kappa_i\eta_i^3\mu_i^4}{400} \cdot \frac{t}{s_{i+1}}.
\end{aligned}$$

This completes the proof. □

Due to the special way in which we chose the sizes of the sets $X_1, X_2, \ldots, X_h$ (specifically, each set X_i is substantially larger than each set X_j for $j > i$) and because each set X_i was computed in the absence of all sets X_j for $j < i$, we can show that each set X_i sends large (cheap and capacitated) flow to S_{i+1} even when all of the other X_j's are removed.

Lemma 6.16. *Consider the settings of an execution of* `ConstructSetsAlg`. *For all $i, j \in \{1, 2, \ldots, h\}$, $i \leq j$, there exist a subset $\widetilde{Z}_{i,j}^\star \subseteq S_{j+1}$ and an $(X_i, \widetilde{Z}_{i,j}^\star)$-flow $(\widetilde{\mathcal{F}}_{i,j}^\star, \widetilde{f}_{i,j}^\star)$ in $G - (X^{<j+1} \setminus X_i)$ with the three following properties where $\ell = \max(i, j)$.*

1. $|\widetilde{Z}_{i,j}^\star| \geq \frac{p_\ell^2\eta_\ell^3\mu_\ell^4}{400} \cdot s_\ell$.
2. *$(\widetilde{\mathcal{F}}_{i,j}^\star, \widetilde{f}_{i,j}^\star)$ is δ-cheap and $\kappa_\ell \cdot \frac{(s_\ell-1)t}{s_\ell^2}$-capacitated.*
3. *For every vertex $z \in \widetilde{Z}_{i,j}^\star$, it holds that* $\sum_{F \in \widetilde{\mathcal{F}}_{i,j}^\star \cap \mathcal{P}^O_{G-(X^{<j+1}\setminus X_i)}(X_i, z)} \widetilde{f}_{i,j}^\star(F) \geq \frac{p_\ell^2\kappa_\ell\eta_\ell^3\mu_\ell^4}{400} \cdot \frac{t}{s_{\ell+1}}$.

Proof. The proof is by induction on j.

Basis. In the basis, $j \leq i$. Let $Z_i^\star \subseteq S_{i+1} \subseteq S_{j+1}$ and $(\mathcal{F}_i^\star, f_i^\star)$ be the subset and $(X_i, Z_i^\star)$-flow (in G_i) whose existence is guaranteed by Lemma 6.15. Then, the following three properties are satisfied.

1. $|Z_i^\star| \geq \frac{p_i^2\eta_i^3\mu_i^4}{400} \cdot s_i$.

2. $(\mathcal{F}_i^\star, f_i^\star)$ is δ-cheap and $\kappa_i \cdot \dfrac{(s_i-1)t}{s_i^2}$-capacitated.

3. For every vertex $z \in Z_i^\star$, it holds that $\displaystyle\sum_{F\in\mathcal{F}_i^\star\cap\mathcal{P}^O_{G_i}(X_i,z)} f_i^\star(F) \geq \frac{p_i^2\kappa_i\eta_i^3\mu_i^4}{400}\cdot\frac{t}{s_{i+1}}$.

Observe that none of the paths in $\mathcal{F}_i^\star$ intersects $X^{<j+1}\setminus X_i$ since $X^{<j+1}\setminus X_i \subseteq X^{<i}$ and $G_i = G - X^{<i}$. Thus, by setting $\widetilde{Z}^\star_{i,j} = Z_i^\star$, and $(\widetilde{\mathcal{F}}^\star_{i,j}, \widetilde{f}^\star_{i,j}) = (\mathcal{F}_i^\star, f_i^\star)$, the proof of the basis is complete.

Step. Now, suppose that the claim holds for $j-1$, and let us prove it for $j \geq i+1$. By the inductive hypothesis, there exist a subset $\widetilde{Z}^\star_{i,j-1} \subseteq S_j$ and an $(X_i, \widetilde{Z}^\star_{i,j-1})$-flow $(\widetilde{\mathcal{F}}^\star_{i,j-1}, \widetilde{f}^\star_{i,j-1})$ in $G-(X^{<j}\setminus X_i)$ with the three following properties.

1. $|\widetilde{Z}^\star_{i,j-1}| \geq \dfrac{p_{j-1}^2\eta_{j-1}^3\mu_{j-1}^4}{400}\cdot s_{j-1}$.

2. $(\widetilde{\mathcal{F}}^\star_{i,j-1}, \widetilde{f}^\star_{i,j-1})$ is δ-cheap and $\kappa_{j-1}\cdot\dfrac{(s_{j-1}-1)t}{s_{j-1}^2}$-capacitated.

3. For every vertex $z \in \widetilde{Z}^\star_{i,j-1}$, $\displaystyle\sum_{F\in\widetilde{\mathcal{F}}^\star_{i,j-1}\cap\mathcal{P}^O_{G-(X^{<j}\setminus X_i)}(X_i,z)} \widetilde{f}^\star_{i,j-1}(F) \geq \frac{p_{j-1}^2\kappa_{j-1}\eta_{j-1}^3\mu_{j-1}^4}{400}\cdot\frac{t}{s_j}$.

Note that the power of $(\widetilde{\mathcal{F}}^\star_{i,j-1}, \widetilde{f}^\star_{i,j-1}(F))$ can be bounded from below as follows.

$$\begin{aligned}\sum_{F\in\widetilde{\mathcal{F}}^\star_{i,j-1}} \widetilde{f}^\star_{i,j-1}(F) &\geq |\widetilde{Z}^\star_{i,j-1}|\cdot\frac{p_{j-1}^2\kappa_{j-1}\eta_{j-1}^3\mu_{j-1}^4}{400}\cdot\frac{t}{s_j}\\ &\geq \frac{p_{j-1}^2\eta_{j-1}^3\mu_{j-1}^4}{400}\cdot s_{j-1}\cdot\frac{p_{j-1}^2\kappa_{j-1}\eta_{j-1}^3\mu_{j-1}^4}{400}\cdot\frac{t}{s_j}\\ &= p_{j-1}^4\cdot\frac{\kappa_{j-1}\eta_{j-1}^6\mu_{j-1}^8}{16\cdot 10^4}\cdot\frac{s_{j-1}}{s_j}\cdot t.\end{aligned}$$

We now modify $\widetilde{f}^\star_{i,j-1}$ to h as follows. Denote $\alpha = \dfrac{\kappa_j\cdot\frac{(s_j-1)t}{s_j^2}}{\kappa_{j-1}\cdot\frac{(s_{j-1}-1)t}{s_{j-1}^2}}$. Then, for every path $F \in \widetilde{\mathcal{F}}^\star_{i,j-1}$, let $h(F) = \alpha\cdot\widetilde{f}^\star_{i,j-1}(F)$. By Observation 6.1, it holds that $\alpha \leq 1$, and therefore $(\widetilde{\mathcal{F}}^\star_{i,j-1}, h)$ is an $(X_i, \widetilde{Z}^\star_{i,j-1})$-flow. Moreover, it is clear that $(\widetilde{\mathcal{F}}^\star_{i,j-1}, h)$ is δ-cheap and that for every vertex $z \in \widetilde{Z}^\star_{i,j-1}$, it holds that $\displaystyle\sum_{F\in\widetilde{\mathcal{F}}^\star_{i,j-1}\cap\mathcal{P}^O_{G-(X^{<j}\setminus X_i)}(X_i,z)} h(F) \geq \alpha\cdot\sum_{F\in\widetilde{\mathcal{F}}^\star_{i,j-1}\cap\mathcal{P}^O_{G-(X^{<j}\setminus X_i)}(X_i,z)} \widetilde{f}^\star_{i,j-1}(F)$. The main property that we have achieved by rescaling that flows is that $(\widetilde{\mathcal{F}}^\star_{i,j-1}, h)$ (unlike necessarily $(\widetilde{\mathcal{F}}^\star_{i,j-1}, \widetilde{f}^\star_{i,j-1})$) is $\kappa_j\cdot\dfrac{(s_j-1)t}{s_j^2}$-capacitated.

In order to proceed with the proof, let us first remark that we can assume without loss of generality that none of the paths in $\widetilde{\mathcal{F}}^\star_{i,j-1}$ contains any vertex from X_i as an internal vertex, because otherwise we can remove the beginning of the paths so that it will still be a path from a vertex in X_i to a vertex in S_j while all of the conditions mentioned earlier will still be satisfied. Moreover, since all of the paths in $\widetilde{\mathcal{F}}^\star_{i,j-1}$ are oriented towards S_j and $S_j \cap X_i = \emptyset$ (because

$S_j \subseteq V(G_j)$ and $j \geq i+1$), we derive that any path in $\widetilde{\mathcal{F}}^\star_{i,j-1}$ only has its first vertex belong to X_i.

Let us denote

$$\mathcal{F} = \{F \in \widetilde{\mathcal{F}}^\star_{i,j-1} : V(F) \cap X_j = \emptyset\}, \text{ and } \overline{\mathcal{F}} = \widetilde{\mathcal{F}}^\star_{i,j-1} \setminus \mathcal{F}.$$

Havin the definitions of $\mathcal{F}$ and $\overline{\mathcal{F}}$ at hand, we first argue that $\sum_{F \in \overline{\mathcal{F}}} h(F) < \mathsf{P}_j + 1$. To this end, targeting a contradiction, suppose that $\sum_{F \in \overline{\mathcal{F}}} h(F) \geq \mathsf{P}_j + 1$. Then, by Lemma 6.10, there exists an (X_j, S_j)-flow of power at least $\mathsf{P}_j + 1$ that is δ-cheap and $\kappa_j \cdot \frac{(s_j-1)t}{s_j^2}$-capacitated. In this case, we can show that $X_j \setminus \{\ell\}$ is a $(\mathsf{P}_j, \delta, \kappa_j \cdot \frac{(s_j-1)t}{s_j^2})$-terminal set in G_j with respect to S_j where ℓ is any leaf of any spanning tree of $G_j[X_j]$ (as in the proof of Lemma 6.11), which contradicts the minimality of X_j. We remark that here, to argue that $X_j \setminus \{\ell\}$ is a $(\mathsf{P}_j, \delta, \kappa_j \cdot \frac{(s_j-1)t}{s_j^2})$-terminal set in G_j rather than only $G - (X^{<j} \setminus X_i)$ (note that $G_j = G - X^{<j}$), we rely on the fact that any path in $\widetilde{\mathcal{F}}^\star_{i,j-1}$ only has its first vertex belong to X_i. Thus, we conclude that

$$\sum_{F \in \overline{\mathcal{F}}} h(F) < \mathsf{P}_j + 1.$$

Therefore, we derive that the power of $(\mathcal{F}, h|_{\mathcal{F}})$, which we denote by P_h, can be bounded from below as follows.

$$\begin{aligned}
\sum_{F \in \mathcal{F}} h(F) &= \sum_{F \in \widetilde{\mathcal{F}}^\star_{i,j-1}} h(F) - \sum_{F \in \overline{\mathcal{F}}} h(F) \\
&> \alpha \cdot \sum_{F \in \widetilde{\mathcal{F}}^\star_{i,j-1}} \widetilde{f}^\star_{i,j-1}(F) - (\mathsf{P}_j + 1) \\
&\geq \frac{\kappa_j \cdot \frac{(s_j-1)t}{s_j^2}}{\kappa_{j-1} \cdot \frac{(s_{j-1}-1)t}{s_{j-1}^2}} \cdot p_{j-1}^4 \frac{\kappa_{j-1}\eta_{j-1}^6 \mu_{j-1}^8}{16 \cdot 10^4} \cdot \frac{s_{j-1}}{s_j} \cdot t - (p_j \kappa_j \eta_j \mu_j^2 \cdot t + 1) \\
&\geq \frac{s_{j-1}^3}{s_j^3} \cdot p_{j-1}^4 \frac{\kappa_j \eta_{j-1}^6 \mu_{j-1}^8}{16 \cdot 10^4} \cdot t - (p_j \kappa_j \eta_j \mu_j^2 \cdot t + 1) \\
&\geq \frac{1}{(1 - 2\mu_{j-1}\sqrt{p_{j-1}})^3} \cdot p_{j-1}^4 \frac{\kappa_j \eta_{j-1}^6 \mu_{j-1}^8}{16 \cdot 10^4} \cdot t - 2p_j \kappa_j \eta_j \mu_j^2 \cdot t \\
&\geq p_{j-1}^4 \frac{\kappa_j \eta_{j-1}^6 \mu_{j-1}^8}{10^6} \cdot t - 2p_j \kappa_j \eta_j \mu_j^2 \cdot t \\
&\geq (p_{j-1}^4 \frac{\eta_{j-1}^5 \mu_{j-1}^6}{10^6} - 2p_j) \cdot \kappa_j \eta_j \mu_j^2 \cdot t.
\end{aligned}$$

Towards the definition of $\widetilde{Z}^\star_{i,j}$, we first define

$$Z_{i,j} = \{z \in \widetilde{Z}^\star_{i,j-1} : \sum_{F \in \mathcal{F} \cap \mathcal{P}^O_{G-(X^{<j+1} \setminus X_i)}(X_i, z)} h(F) \geq \mathsf{P}_h / (2|\widetilde{Z}^\star_{i,j-1}|)\}.$$

By Lemma 6.14, we have that $|Z_{i,j}| \geq \mathsf{P}_h / (2\kappa_j \cdot \frac{(s_j-1)t}{s_j^2})$. Now, we define

$$\widetilde{Z}^\star_{i,j} = Z_{i,j} \cap S_{j+1}.$$

Additionally, we define $\widetilde{\mathcal{F}}^\star_{i,j} = \mathcal{F} \cap \mathcal{P}^O_{G-(X^{<j+1}\setminus X_i)}(X_i, \widetilde{Z}^\star_{i,j})$, and $\widetilde{f}^\star_{i,j} = h|_{\widetilde{\mathcal{F}}^\star_{i,j}}$.

We proceed to verify that $\widetilde{Z}^\star_{i,j}$ and $(\widetilde{\mathcal{F}}^\star_{i,j}, \widetilde{f}^\star_{i,j})$ satisfy the three conditions in the lemma. First, because $(\mathcal{F}, h|_{\mathcal{F}})$ is an $(X_i, \widetilde{Z}^\star_{i,j-1})$-flow in $G - (X^{<j+1} \setminus X_i)$ that is δ-cheap and $\kappa_\ell \cdot \dfrac{(s_\ell - 1)t}{s_\ell^2}$-capacitated, it is immediate that $(\widetilde{\mathcal{F}}^\star_{i,j}, \widetilde{f}^\star_{i,j})$ is $(X_i, \widetilde{Z}^\star_{i,j})$-flow in $G - (X^{<j+1} \setminus X_i)$ that is δ-cheap and $\kappa_\ell \cdot \dfrac{(s_\ell - 1)t}{s_\ell^2}$-capacitated. In particular, Condition 2 is satisfied.

For Condition 3, note that be the definition of $Z_{i,j}$, for every vertex $z \in \widetilde{Z}^\star_{i,j}$, it holds that

$$\begin{aligned}
\sum_{F \in \mathcal{F} \cap \mathcal{P}^O_{G-(X^{<j+1}\setminus X_i)}(X_i, z)} h(F) &\geq \mathsf{P}_h / (2|\widetilde{Z}^\star_{i,j-1}|) \\
&\geq \left((p_{j-1}^4 \frac{\eta_{j-1}^5 \mu_{j-1}^6}{10^6} - 2p_j) \cdot \kappa_j \eta_j \mu_j^2 \cdot t \right) / \left(\frac{p_{j-1}^2 \eta_{j-1}^3 \mu_{j-1}^4}{400} \cdot s_{j-1} \right) \\
&\geq \frac{16 \cdot 10^4 \cdot (p_{j-1}^4 \frac{\eta_{j-1}^5 \mu_{j-1}^6}{10^6} - 2p_j)}{p_{j-1}^2 \eta_{j-1}^3 \mu_{j-1}^4 \cdot p_j^2 \eta_j^2 \mu_j^2} \cdot \frac{p_j^2 \kappa_j \eta_j^3 \mu_j^4}{400} \cdot \frac{t}{s_{j-1}} \\
&\geq (1 - 2\mu_i \sqrt{p_i})^2 \cdot \frac{16 \cdot (p_{j-1}^4 \frac{\eta_{j-1}^5 \mu_{j-1}^6}{100} - 10^5 p_j)}{p_{j-1}^2 \eta_{j-1}^3 \mu_{j-1}^4 \cdot p_j^2 \eta_j^2 \mu_j^2} \cdot \frac{p_j^2 \kappa_j \eta_j^3 \mu_j^4}{400} \cdot \frac{t}{s_{j+1}} \\
&\geq \frac{p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 - 10^7 p_j}{100 \cdot p_{j-1}^2 \eta_{j-1}^3 \mu_{j-1}^4 \cdot p_j^2 \eta_j^2 \mu_j^2} \cdot \frac{p_j^2 \kappa_j \eta_j^3 \mu_j^4}{400} \cdot \frac{t}{s_{j+1}}.
\end{aligned}$$

Thus, to conclude the proof of Condition 3, it suffices to show that $\dfrac{p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 - 10^7 p_j}{100 \cdot p_{j-1}^2 \eta_{j-1}^3 \mu_{j-1}^4 \cdot p_j^2 \eta_j^2 \mu_j^2} \geq 1$. To this end, notice that $p_j \leq p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 / (2 \cdot 10^7)$. Thus, $\dfrac{p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 - 10^7 p_j}{100 \cdot p_{j-1}^2 \eta_{j-1}^3 \mu_{j-1}^4 \cdot p_j^2 \eta_j^2 \mu_j^2} \geq \dfrac{p_{j-1}^2 \eta_{j-1}^2 \mu_{j-1}^2}{200 p_j^2 \eta_j^2 \mu_j^2} \geq \dfrac{p_{j-1^2}}{200 p_j^2} \geq 1$.

It remains to prove that Condition 1 is satisfied. That is, we need to show that $|\widetilde{Z}^\star_{i,j}| \geq \dfrac{p_j^2 \eta_j^3 \mu_j^4}{400} \cdot s_j$. To this end, notice that $|\widetilde{Z}^\star_{i,j}| \geq |Z_{i,j}| - (s_j - s_{j+1})$. Now, recall that $s_{j+1} \geq (1 - 2\mu_j \sqrt{p_j}) s_j$. Thus, $|\widetilde{Z}^\star_{i,j}| \geq |Z_{i,j}| - 2\mu_j \sqrt{p_j} s_j$. Next, recall that $|Z_{i,j}| \geq \mathsf{P}_h / (2\kappa_j \cdot \frac{(s_j - 1)t}{s_j^2})$. Therefore, we have that

$$\begin{aligned}
|\widetilde{Z}^\star_{i,j}| &\geq \mathsf{P}_h / (2\kappa_j \cdot \tfrac{(s_j - 1)t}{s_j^2}) - 2\mu_j \sqrt{p_j} s_j \\
&\geq \left((p_{j-1}^4 \tfrac{\eta_{j-1}^5 \mu_{j-1}^6}{10^6} - 2p_j) \cdot \kappa_j \eta_j \mu_j^2 \cdot t \right) / \left(2\kappa_j \cdot \tfrac{(s_j - 1)t}{s_j^2} \right) - 2\mu_j \sqrt{p_j} s_j \\
&\geq \left(p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 \cdot \eta_j \mu_j^2 \right) / \left(10^7 \cdot \tfrac{(s_j - 1)}{s_j^2} \right) - 2\mu_j \sqrt{p_j} s_j \\
&\geq \left(\tfrac{p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 \cdot \eta_j \mu_j^2}{10^7} - 2\sqrt{p_j} \right) \cdot s_j.
\end{aligned}$$

Notice that $p_j \le \frac{p_{j-1}^8 \eta_{j-1}^{10} \mu_{j-1}^{12} \cdot \eta_j^2 \mu_j^4}{10^{20}} \le \frac{1}{4}\left(\frac{p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 \cdot \eta_j \mu_j^2}{10^7}\right)^2$. Thus,

$$\begin{aligned}
|\widetilde{Z}_{i,j}^{\star}| &\ge \frac{p_{j-1}^4 \eta_{j-1}^5 \mu_{j-1}^6 \cdot \eta_j \mu_j^2}{2 \cdot 10^7} \cdot s_j \\
&\ge \frac{p_{j-1}^4}{p_j^2} \cdot \frac{\eta_{j-1}^3 \mu_{j-1}^4}{10^5} \cdot \frac{p_j^2 \eta_j^3 \mu_j^4}{400} \cdot s_j \\
&\ge \frac{p_j^2 \eta_j^3 \mu_j^4}{400} \cdot s_j.
\end{aligned}$$

This completes the proof. □

As a corollary of Lemma 6.16, we derive the following result.

Corollary 6.8. *Consider the settings of an execution of* `ConstructSetsAlg`*. Suppose that $t \ge 1/(p_h \kappa_h \eta_h \mu_h^2)$. For any $i \in \{1, 2, \dots, h\}$, there exist a subset $\widetilde{Z}_i^{\star} \subseteq S_{h+1}$ and an $(X_i, \widetilde{Z}_i^{\star})$-flow $(\widetilde{\mathcal{F}}_i^{\star}, \widetilde{f}_i^{\star})$ in $G - \overline{X}_i$ with the three following properties.*

1. $|\widetilde{Z}_i^{\star}| \ge \dfrac{p_h^2 \eta_h^3 \mu_h^4}{400} \cdot s_h$.
2. *$(\widetilde{\mathcal{F}}_i^{\star}, \widetilde{f}_i^{\star})$ is δ-cheap and $\kappa_h \cdot \dfrac{(s_h - 1)t}{s_h^2}$-capacitated.*
3. *For every vertex $z \in \widetilde{Z}_i^{\star}$, it holds that* $\displaystyle\sum_{F \in \widetilde{\mathcal{F}}_i^{\star} \cap \mathcal{P}^O_{G - \overline{X}_i}(X_i, z)} \widetilde{f}_i^{\star}(F) \ge \frac{p_h^2 \kappa_h \eta_h^3 \mu_h^4}{400} \cdot \frac{t}{s_{h+1}}$.

Furthermore, we show that also large enough subsets of each set S_h can send to each other large (cheap and capacitated) flow to S_{h+1} even when all of the X_i's are removed. Here we will make use of Corollary 6.6.

Lemma 6.17. *Let $0 < \gamma < 1$. Consider the settings of an execution of* `ConstructSetsAlg`*. For any disjoint subsets $A, B \subseteq S_{h+1}$ such that $|A|, |B| \ge \gamma s_{h+1}$, there exists an (A, B)-flow $(\mathcal{F}_{A,B}, f_{A,B})$ in $G - X^{\star}$ with the four following properties.*

1. *The power of $(\mathcal{F}_{A,B}, f_{A,B})$ is at least $\kappa_{h+1}(\dfrac{\gamma \eta_{h+1}}{4})^{\frac{4}{\gamma \eta_{h+1}}+1} \cdot t$.*
2. *For every vertex $a \in A$, it holds that* $\displaystyle\sum_{F \in \mathcal{F}_{A,B} \cap \mathcal{P}^O_{G - X^{\star}}(a, B)} f_{A,B}(F) \le \kappa_{h+1} \cdot (t/s_{h+1})$.
3. *For every vertex $b \in B$, it holds that* $\displaystyle\sum_{F \in \mathcal{F}_{A,B} \cap \mathcal{P}^O_{G - X^{\star}}(A, b)} f_{A,B}(F) \le \kappa_{h+1} \cdot (t/s_{h+1})$.
4. *$(\mathcal{F}_{A,B}, f_{A,B})$ is $\dfrac{4\delta}{\gamma \cdot \eta_{h+1}}$-cheap.*

Proof. Let $A, B \subseteq S_{h+1}$ be such that $|A|, |B| \ge \gamma s_{h+1}$. Recall that $(S_{h+1}, H_{h+1}, \mathsf{Q}_{h+1})$ is a nice $(s_{h+1}, t, \delta, \kappa_{h+1}, \eta_{h+1})$-witness with respect to (G_{h+1}, x_{h+1}). By Corollary 6.6, it is also $(\gamma, \rho, \nu, \zeta)$-useful where $\rho = \dfrac{\gamma \eta_{h+1}}{2}$, $\zeta = (\dfrac{2}{\rho})^{\nu - 1}$ and $\nu = \dfrac{4}{\gamma \eta_{h+1}}$.

By Definition 6.6, there is a set of paths $\mathcal{P}^{\star} \subseteq \mathcal{P}_{H_{h+1}}(A, B)$ with the following properties.

- Any path in $\mathcal{P}^{\star}$ contains at most $\dfrac{4}{\gamma \cdot \eta_{h+1}}$ edges.

- $|\mathcal{P}^\star| \geq \frac{\gamma\rho}{2} \cdot s_{h+1}^2 = \frac{\gamma^2 \eta_{h+1}}{4} \cdot s_{h+1}^2$.
- For any edge $e \in E(H_{h+1})$, it holds that $|\{P \in \mathcal{P}^\star : e \in E(P)\}| \leq (\dfrac{2}{\rho})^{\nu-1} \leq (\dfrac{4}{\gamma\eta_{h+1}})^{\frac{4}{\gamma\eta_{h+1}}}$.

Because $(S_{h+1}, H_{h+1}, \mathsf{Q}_{h+1})$ is an $(s_{h+1}, t, \delta, \kappa_{h+1}, \eta_{h+1})$-witness, the following condition holds: for every edge $e \in E(H_{h+1})$, it holds that $\kappa_{h+1} \cdot (t/s_{h+1}^2) = \sum_{Q \in \mathcal{Q}_e} \lambda_e(Q)$ where $\mathcal{Q}_e \in \mathsf{Q}_{h+1}$. Therefore, for every path $P^\star \in \mathcal{P}^\star$ with endpoints $u \in A$ and $v \in B$, there exist a set of walks $\mathcal{W}_{P^\star} \subseteq \mathcal{W}_{G_{h+1}}(u, v)$ and a function $f_{P^\star} : \mathcal{W}_{P^\star} \to \mathbb{Q}^+$ such that

- $\displaystyle\sum_{W \in \mathcal{W}_{P^\star}} f_{P^\star}(W) = \kappa_{h+1} \cdot (t/s_{h+1}^2)$, and
- for every vertex $w \in V(G_{h+1})$, $\displaystyle\sum_{\substack{W \in \mathcal{W}_{P^\star} \\ \text{s.t. } w \in V(W)}} f_{P^\star}(W) \leq \sum_{e \in E(P)} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } w \in V(Q)}} \lambda_e(Q)$. Here, we use the supposition that $A \cap B = \emptyset$ since then every path in $\mathcal{P}^\star$ has at least one edge, which ensures that the "contribution" of its endpoints is taken into account.

Since every walk contains a path with the same endpoints, for every path $P \in \mathcal{P}^\star$ with endpoints $u \in A$ and $v \in B$, there exist a set of paths $\widehat{\mathcal{P}}_{P^\star} \subseteq \mathcal{P}_{G_{h+1}}(u, v)$ and a function $\widehat{f}_{P^\star} : \widehat{\mathcal{P}}_{P^\star} \to \mathbb{Q}^+$ such that

- $\displaystyle\sum_{P \in \widehat{\mathcal{P}}_{P^\star}} \widehat{f}_{P^\star}(P) = \kappa_{h+1} \cdot (t/s_{h+1}^2)$, and
- for every vertex $w \in V(G_{h+1})$, $\displaystyle\sum_{\substack{P \in \widehat{\mathcal{P}}_{P^\star} \\ \text{s.t. } w \in V(P)}} \widehat{f}_{P^\star}(P) \leq \sum_{e \in E(P^\star)} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } w \in V(Q)}} \lambda_e(Q)$.

We define $\mathcal{F}_{A,B} = \bigcup_{P^\star \in \mathcal{P}^\star} \widehat{\mathcal{P}}_{P^\star}$. For every path $F \in \mathcal{F}_{A,B}$, let $g(F)$ denote the set of paths $P^\star \in \mathcal{P}^\star$ such that $F \in \widehat{\mathcal{P}}_{P^\star}$, and define

$$f_{A,B}(F) = \alpha \cdot \sum_{P^\star \in g(F)} \widehat{f}_{P^\star}(F), \text{ where } \alpha = (\frac{\gamma\eta_{h+1}}{4})^{\frac{4}{\gamma\eta_{h+1}}}.$$

Let us first show that $(\mathcal{F}_{A,B}, f_{A,B})$ is an (A, B)-flow. Because $V(G_{h+1}) \cap X^\star = \emptyset$, is clear that every path in $\mathcal{F}_{A,B}$ belongs to $\mathcal{P}_{G-X^\star}(A, B)$. Moreover, for every vertex $v \in V(G_{h+1})$, it holds that

$$\begin{aligned}
\sum_{\substack{F \in \mathcal{F}_{A,B} \\ \text{s.t. } v \in V(F)}} f_{A,B}(F) &\leq \alpha \cdot \sum_{\substack{F \in \mathcal{F}_{A,B} \\ \text{s.t. } v \in V(F)}} \sum_{P^\star \in g(F)} \widehat{f}_{P^\star}(F) \\
&= \alpha \cdot \sum_{P^\star \in \mathcal{P}^\star} \sum_{\substack{P \in \mathcal{P}_{P^\star} \\ \text{s.t. } v \in V(P)}} \widehat{f}_{P^\star}(F) \\
&\leq \alpha \cdot \sum_{P^\star \in \mathcal{P}^\star} \sum_{e \in E(P^\star)} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } v \in V(Q)}} \lambda_e(Q) \\
&= \alpha \cdot \sum_{e \in E(H_{h+1})} \sum_{\substack{P^\star \in \mathcal{P}^\star \\ \text{s.t. } e \in E(P^\star)}} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } v \in V(Q)}} \lambda_e(Q) \\
&\leq \alpha \cdot (\frac{4}{\gamma\eta_{h+1}})^{\frac{4}{\gamma\eta_{h+1}}} \cdot \sum_{e \in E(H_{h+1})} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } v \in V(Q)}} \lambda_e(Q) \\
&= \sum_{e \in E(H_{h+1})} \sum_{\substack{Q \in \mathcal{Q}_e \\ \text{s.t. } v \in V(Q)}} \lambda_e(Q) \leq 1.
\end{aligned}$$

Let us now show that the four properties in the lemma are satisfied. For the first property, note that the power of $(\mathcal{F}_{A,B}, f_{A,B})$ can be lower bounded as follows.

$$\begin{aligned}
\sum_{F\in\mathcal{F}_{A,B}} f_{A,B}(F) &= \alpha\cdot\sum_{F\in\mathcal{F}_{A,B}}\sum_{P^\star\in g(F)} \widehat{f}_{P^\star}(F)\\
&= \alpha\cdot\sum_{P^\star\in\mathcal{P}^\star}\sum_{F\in\widehat{\mathcal{P}}_{P^\star}} \widehat{f}_{P^\star}(F)\\
&= \alpha\cdot\sum_{P^\star\in\mathcal{P}^\star} \kappa_{h+1}\cdot(t/s_{h+1}^2)\\
&= (\frac{\gamma\eta_{h+1}}{4})^{\frac{4}{\gamma\eta_{h+1}}}\cdot|\mathcal{P}^\star|\cdot\kappa_{h+1}\cdot(t/s_{h+1}^2)\\
&\geq (\frac{\gamma\eta_{h+1}}{4})^{\frac{4}{\gamma\eta_{h+1}}}\cdot\frac{\gamma^2\eta_{h+1}}{4}\cdot s_{h+1}^2\cdot\kappa_{h+1}\cdot(t/s_{h+1}^2)\\
&\geq \kappa_{h+1}(\frac{\gamma\eta_{h+1}}{4})^{\frac{4}{\gamma\eta_{h+1}}+1}\cdot t.
\end{aligned}$$

For the fourth property, recall that every path in $\mathcal{P}^\star$ contains at most $\dfrac{4}{\gamma\cdot\eta_{h+1}}$ edges. This means that every path in $\mathcal{F}_{A,B}$ is the concatenation of at most $\dfrac{4}{\gamma\cdot\eta_{h+1}}$ paths from $\bigcup \mathsf{Q}_{h+1}$. By Condition 1 in Definition 6.1, for every edge $e\in E(H)$ and $P\in\mathcal{Q}_e$, it holds that $\sum_{v\in V(P)} x_v\leq\delta$. From this, we conclude that $(\mathcal{F}_{A,B}, f_{A,B})$ is $\dfrac{4\delta}{\gamma\cdot\eta_{h+1}}$-cheap.

It remains to prove the the second and third properties are satisfied. Since their proofs are symmetric, we only show that the second property is satisfied. To this end, consider some vertex $a\in A$. Then, we need to show that $\displaystyle\sum_{F\in\mathcal{F}_{A,B}\cap\mathcal{P}^O_{G-X^\star}(a,B)} f_{A,B}(F)\leq\kappa_{h+1}\cdot(t/s_{h+1})$. For this purpose, note that

$$\begin{aligned}
\sum_{F\in\mathcal{F}_{A,B}\cap\mathcal{P}^O_{G-X^\star}(a,B)} f_{A,B}(F) &= \alpha\cdot\sum_{F\in\mathcal{F}_{A,B}\cap\mathcal{P}^O_{G-X^\star}(a,B)}\sum_{P^\star\in g(F)}\widehat{f}_{P^\star}(F)\\
&= \alpha\cdot\sum_{P^\star\in\mathcal{P}^\star\cap\mathcal{P}_{H_{h+1}}(a,B)}\sum_{P\in\mathcal{P}_{P^\star}}\widehat{f}_{P^\star}(F)\\
&\leq \alpha\cdot\sum_{\substack{e\in E(H_{h+1})\\ \text{s.t. } a\in e}}\sum_{\substack{P^\star\in\mathcal{P}^\star\\ \text{s.t. } e\in E(P^\star)}}\sum_{P\in\mathcal{P}_{P^\star}}\widehat{f}_{P^\star}(F)\\
&\leq \alpha\cdot\sum_{\substack{e\in E(H_{h+1})\\ \text{s.t. } a\in e}}\sum_{\substack{P^\star\in\mathcal{P}^\star\\ \text{s.t. } e\in E(P^\star)}}\kappa_{h+1}\cdot(t/s_{h+1}^2)\\
&\leq (\frac{\gamma\eta_{h+1}}{4})^{\frac{4}{\gamma\eta_{h+1}}}\cdot\sum_{\substack{e\in E(H_{h+1})\\ \text{s.t. } a\in e}}(\frac{4}{\gamma\eta_{h+1}})^{\frac{4}{\gamma\eta_{h+1}}}\cdot\kappa_{h+1}\cdot(t/s_{h+1}^2)\\
&\leq (\frac{\gamma\eta_{h+1}}{4})^{\frac{4}{\gamma\eta_{h+1}}}\cdot s_{h+1}\cdot(\frac{4}{\gamma\eta_{h+1}})^{\frac{4}{\gamma\eta_{h+1}}}\cdot\kappa_{h+1}\cdot(t/s_{h+1}^2)\\
&= \kappa_{h+1}\cdot(t/s_{h+1}).
\end{aligned}$$

This completes the proof. □

Let us extend Lemma 6.17 to the case where A and B may not be disjoint.

Lemma 6.18. *Let $0<\gamma<1$. Consider the settings of an execution of* `ConstructSetsAlg`. *For any subsets $A,B\subseteq S_{h+1}$ such that $|A|,|B|\geq\gamma s_{h+1}$, there exists an (A,B)-flow $(\mathcal{F}_{A,B}, f_{A,B})$ in $G-X^\star$ with the four following properties.*

1. *The power of $(\mathcal{F}_{A,B}, f_{A,B})$ is at least $\kappa_{h+1}(\frac{\gamma\eta_{h+1}}{8})^{\frac{8}{\gamma\eta_{h+1}}+1}\cdot t$.*

2. *For every vertex $a \in A$, it holds that* $\displaystyle\sum_{F \in \mathcal{F}_{A,B} \cap \mathcal{P}^O_{G-X^\star}(a,B)} f_{A,B}(F) \leq \kappa_{h+1} \cdot (t/s_{h+1})$.

3. *For every vertex $b \in B$, it holds that* $\displaystyle\sum_{F \in \mathcal{F}_{A,B} \cap \mathcal{P}^O_{G-X^\star}(A,b)} f_{A,B}(F) \leq \kappa_{h+1} \cdot (t/s_{h+1})$.

4. *$(\mathcal{F}_{A,B}, f_{A,B})$ is* $\dfrac{8\delta}{\gamma \cdot \eta_{h+1}}$*-cheap.*

Proof. We consider two cases as follows. In the first case, suppose that $|A \cap B| \leq \frac{\gamma}{2} s_{h+1}$. Then, lemma follows by a direct application of Lemma 6.17 with $A' = A \setminus B$, $B' = B \setminus A$ and $\gamma' = \gamma/2$.

In the second case, suppose that $|A \cap B| \geq \frac{\gamma}{2} s_{h+1}$. Then, let $\mathcal{F}_{A,B}$ be the set of every path that consists of only one vertex and this single vertex belongs to $A \cap B$. For every path $F \in \mathcal{F}_{A,B}$, define $f_{A,B}(F) = \kappa_{h+1} \cdot (t/s_{h+1})$. Then, it is immediate that $(\mathcal{F}_{A,B}, f_{A,B})$ is an (A,B)-flow in $G - X^\star$ and that the second and third properties in the lemma are satisfied. For the first property, note that the power of $(\mathcal{F}_{A,B}, f_{A,B})$ is exactly $|A \cap B| \cdot \kappa_{h+1} \cdot (t/s_{h+1})$, which is lower bounded by $\kappa_{h+1} \frac{\gamma}{2} \cdot t$. Thus, the first property is satisfied as well.

Finally, for the fourth property, it suffice to show that $x_v \leq \frac{8\delta}{\gamma \cdot \eta_{h+1}}$ for every vertex $v \in A \cap B$. Recall that $(S_{h+1}, H_{h+1}, \mathsf{Q}_{h+1})$ is a nice $(s_{h+1}, t, \delta, \kappa_{h+1}, \eta_{h+1})$-witness with respect to (G_{h+1}, x_{h+1}). By Corollary 6.6, it is also $(\gamma, \rho, \nu, \zeta)$-useful where $\rho, \nu, \zeta > 0$. In particular, this means that there exists an edge $e \in E(H_{h+1})$ that is incident to v. Moreover, because $\kappa \cdot (t/s^2) = \sum_{P \in \mathcal{Q}_e} \lambda_e(P)$, we have that there exists at least one path P in $\mathcal{Q}_e$. Then, it holds that $\sum_{v \in V(P)} x_v \leq \delta$. In particular, this means that $x_v \leq \delta$ (which is upper bounded by $\frac{8\delta}{\gamma \cdot \eta_{h+1}}$). This completes the proof. □

Now, in order to send large flow from some set X_i to some set X_j while avoiding all of the other X_t's, we are going to send flow first *(i)* from X_i to $\widetilde{Z}^\star_i \subseteq S_{h+1}$, then *(ii)* from $\widetilde{Z}^\star_i$ to $\widetilde{Z}^\star_j \subseteq S_{h+1}$, and finally *(iii)* from $\widetilde{Z}^\star_j$ to X_j. To this end, we will make use of Corollary 6.8 and Lemma 6.18. In particular, we need to "connect" the paths realizing the three different parts above. This goal is achieved in the following lemma, which is the main result of this section. This lemma will be the crux of the translation ahead.

Lemma 6.19. *Let H be a graph with $|V(H)| = h$. There exists a fixed constant $d = d(H) > 0$ such that for any instance of* **Well-Linkedness LP**, *say* **Well-Linkedness LP**(G, w, t)*, where G is H-minor free and $t \geq 1/d^5$, and for any assignment $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}^+_0$ that cannot be extended to a feasible fractional solution, the following condition is satisfied:*

> *There exists $\overline{X} \in \mathtt{ConPart}(G, h)$ such that for all distinct $i, j \in \{1, 2, \ldots, h\}$, there exists an $(\overline{X}[i], \overline{X}[j])$-flow $(\mathcal{F}_{i,j}, f_{i,j})$ in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ (where $X = \bigcup_{\ell=1}^{h} \overline{X}[\ell]$) of power at least $(\frac{d^{11}}{10^4})^{\frac{10^4}{d^{10}}+2} \cdot t$ and which is $\frac{10^4}{d^{10}} + 4$-cheap.*

Proof. Notice that there exists a fixed constant $0 < d = d(H) < 1$ such that for any instance of **Well-Linkedness LP**, say **Well-Linkedness LP**(G, w, t), where G is H-minor free, and for any assignment $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}^+_0$ that cannot be extended to a feasible fractional solution, it holds that $d \leq \min\{p_h, \kappa_h, \eta_h, \mu_h\}$ where $p_h, \kappa_h, \eta_h, \mu_h$ are the values that arise in the execution of `ConstructSetsAlg` with respect to this input. In what follows, we work with the constant d, and consider some instance of **Well-Linkedness LP**, say **Well-Linkedness LP**(G, w, t), where G is H-minor free and $t \geq 1/d^5$. Moreover, we consider some assignment $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}^+_0$ that cannot be extended to a feasible fractional solution. Consider an execution of `ConstructSetsAlg` with respect to this input. In what follows, we use the notation corresponding to this execution. In particular, let $X_1, X_2, \dots, X_h$ and $S_1, S_2, \dots, S_h$ be its output.

Notice that $t \geq 1/d^5 \geq 1/(p_h\kappa_h\eta_h\mu_h^2)$. Then, by Corollary 6.8, there exist a subset $\widetilde{Z}_i^\star \subseteq S_{h+1}$ and an $(X_i, \widetilde{Z}_i^\star)$-flow $(\widetilde{\mathcal{F}}_i^\star, \widetilde{f}_i^\star)$ in $G - \overline{X}_i$ with the three following properties.

1. $|\widetilde{Z}_i^\star| \geq \dfrac{p_h^2\eta_h^3\mu_h^4}{400} \cdot s_h \geq \dfrac{d^9}{400} \cdot s_h$.
2. $(\widetilde{\mathcal{F}}_i^\star, \widetilde{f}_i^\star)$ is 2-cheap.
3. For every vertex $z \in \widetilde{Z}_i^\star$, $\displaystyle\sum_{F \in \widetilde{\mathcal{F}}_i^\star \cap \mathcal{P}^O_{G-\overline{X}_i}(X_i,z)} \widetilde{f}_i^\star(F) \geq \frac{p_h^2\kappa_h\eta_h^3\mu_h^4}{400} \cdot \frac{t}{s_{h+1}} \geq \frac{d^{10}}{400} \cdot \frac{t}{s_{h+1}}$.

Moreover, there exist a subset $\widetilde{Z}_j^\star \subseteq S_{h+1}$ and an $(X_j, \widetilde{Z}_j^\star)$-flow $(\widetilde{\mathcal{F}}_j^\star, \widetilde{f}_j^\star)$ in $G - \overline{X}_j$ that satisfies the three properties analogous to those above.

By Lemma 6.18, there exists a $(\widetilde{Z}_i^\star, \widetilde{Z}_j^\star)$-flow $(\widehat{\mathcal{F}}_{i,j}, \widehat{f}_{i,j})$ in $G - X^\star$ with the four following properties with $\gamma = \frac{d^9}{400}$.

1. The power of $(\widehat{\mathcal{F}}_{i,j}, \widehat{f}_{i,j})$ is at least $\kappa_{h+1}(\dfrac{\gamma\eta_{h+1}}{8})^{\frac{8}{\gamma\eta_{h+1}}+1} \cdot t \geq (\dfrac{d^{11}}{10^4})^{\frac{10^4}{d^{10}}+1} \cdot t$.
2. For every vertex $a \in \widetilde{Z}_i^\star$, it holds that $\displaystyle\sum_{F \in \widehat{\mathcal{F}}_{i,j} \cap \mathcal{P}^O_{G-X^\star}(a,\widetilde{Z}_j^\star)} \widehat{f}_{i,j}(F) \leq \kappa_{h+1} \cdot (t/s_{h+1})$.
3. For every vertex $b \in \widetilde{Z}_j^\star$, it holds that $\displaystyle\sum_{F \in \widehat{\mathcal{F}}_{i,j} \cap \mathcal{P}^O_{G-X^\star}(\widetilde{Z}_i^\star,b)} \widehat{f}_{i,j}(F) \leq \kappa_{h+1} \cdot (t/s_{h+1})$.
4. $(\widehat{\mathcal{F}}_{i,j}, \widehat{f}_{i,j})$ is $\dfrac{16}{\gamma \cdot \eta_{h+1}}$-cheap. Thus, it is $\dfrac{10^4}{d^{10}}$-cheap.

We define $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ by setting $\widehat{\mathcal{F}}_{i,j}^\star = \widehat{\mathcal{F}}_{i,j}$ and $\widehat{f}_{i,j}^\star(F) = \dfrac{d^{10}}{400 \cdot \kappa_{h+1}} \cdot \widehat{f}_{i,j}(F)$ for all $F \in \widehat{\mathcal{F}}_{i,j}^\star$. Note that $\frac{d^{10}}{400 \cdot \kappa_{h+1}} \leq 1$, and hence $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ is a $(\widetilde{Z}_i^\star, \widetilde{Z}_j^\star)$-flow. Furthermore, it satisfies the following properties.

1. The power of $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ is at least $\dfrac{d^{10}}{400 \cdot \kappa_{h+1}} \cdot (\dfrac{d^{11}}{10^4})^{\frac{10^4}{d^{10}}+1} \cdot t \geq (\dfrac{d^{11}}{10^4})^{\frac{10^4}{d^{10}}+2} \cdot t$.
2. For every vertex $a \in \widetilde{Z}_i^\star$, it holds that $\displaystyle\sum_{F \in \widehat{\mathcal{F}}_{i,j}^\star \cap \mathcal{P}^O_{G-X^\star}(a,\widetilde{Z}_j^\star)} \widehat{f}_{i,j}^\star(F) \leq \frac{d^{10}}{400} \cdot \frac{t}{s_{h+1}}$.
3. For every vertex $b \in \widetilde{Z}_j^\star$, it holds that $\displaystyle\sum_{F \in \widehat{\mathcal{F}}_{i,j}^\star \cap \mathcal{P}^O_{G-X^\star}(\widetilde{Z}_i^\star,b)} \widehat{f}_{i,j}^\star(F) \leq \frac{d^{10}}{400} \cdot \frac{t}{s_{h+1}}$.
4. $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ is $\dfrac{10^4}{d^{10}}$-cheap.

Having $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$, $(\widetilde{\mathcal{F}}_i^\star, \widetilde{f}_i^\star)$ and $(\widetilde{\mathcal{F}}_j^\star, \widetilde{f}_j^\star)$ at hand, we can construct an (X_i, X_j)-flow $(\mathcal{F}_{i,j}, f_{i,j})$ in $G - (X^\star \setminus (X_i \cup X_j))$ of the same power as $(\widehat{\mathcal{F}}_{i,j}, \widehat{f}_{i,j})$ and whose cheapness is upper bounded the summation of the cheapness of$(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$, $(\widetilde{\mathcal{F}}_i^\star, \widetilde{f}_i^\star)$ and $(\widetilde{\mathcal{F}}_j^\star, \widetilde{f}_j^\star)$. To see this, first use the entire power of the flow $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ to transmit flow between $\widetilde{Z}_i^\star$ and $\widetilde{Z}_j^\star$. Now, recall that for every vertex $z \in \widetilde{Z}_i^\star$, $\sum_{F \in \widetilde{\mathcal{F}}_i^\star \cap \mathcal{P}^O_{G-\overline{X}_i}(X_i,z)} \widetilde{f}_i^\star(F) \geq \frac{d^{10}}{400} \cdot \frac{t}{s_{h+1}}$. Thus, we can use (part of the power of) the flow $(\widetilde{\mathcal{F}}_i^\star, \widetilde{f}_i^\star)$ to send *all* of the flow transmitted by $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ to $\widetilde{Z}_i^\star$ back to X_i. Symmetrically, we can use (part of the power of) the flow $(\widetilde{\mathcal{F}}_j^\star, \widetilde{f}_j^\star)$ to send *all* of the flow transmitted by $(\widehat{\mathcal{F}}_{i,j}^\star, \widehat{f}_{i,j}^\star)$ to $\widetilde{Z}_j^\star$ back to X_j. Since every walk contains a path with the same endpoints, this results in an (X_i, X_j)-flow $(\mathcal{F}_{i,j}, f_{i,j})$ as required. □

Translation. Finally, we present the translation. Here, given a feasible fractional solution of **Pairwise-Flow Hitting LP**, the translation entails the multiplication of the value assigned to each variable x_v by a fixed constant, along with the extension of the result to the variable set of **Well-Linkedness LP**. We also pay the penalty of multiplying t by a fixed constant.

For the proof of the correctness of our translation, we need the following lemma, which asserts that a cheap (A,B)-flow of large power witnesses that **Penalized Flow Packing LP**(G,x,A,B) has a large optimum.

Lemma 6.20. *Let* **Penalized Flow Packing LP**(G,x,A,B) *be an instance of* **Penalized Flow Packing LP**. *Suppose that G has an (A,B)-flow $(\mathcal{F},f)$ of power larger than $h(h-1)t$ and which is $1/2$-cheap. Then, the optimum of* **Penalized Flow Packing LP**(G,x,A,B) *is larger than $\frac{h(h-1)}{2}t$.*

Proof. With respect to **Penalized Flow Packing LP**(G,x,A,B), we define an assignment $\alpha : \{z_P : P \in \mathcal{P}_G(A,B)\} \to \mathbb{Q}_0^+$ as follows. For every $P \in \mathcal{P}_G(A,B)$,

- if $P \in \mathcal{F}$, then $\alpha(z_P) = f(P)$, and
- otherwise, $\alpha(z_P) = 0$.

We now argue that α is a feasible solution for **Penalized Flow Packing LP**(G,x,A,B). To show this, consider some vertex $v \in V(G)$. Then, we need to show that α satisfies the constraint $\sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} z_P \leq 1$. For this purpose, note that

$$\sum_{\substack{P \in \mathcal{P}(A,B) \\ \text{s.t. } v \in V(P)}} \alpha(z_P) \leq \sum_{\substack{F \in \mathcal{F} \\ \text{s.t. } v \in V(P)}} f(F) \leq 1.$$

Here, the last inequality follows from the fact that $(\mathcal{F},f)$ is a valid flow.

Because α is a feasible solution for **Penalized Flow Packing LP**(G,x,A,B), to conclude the proof, it suffices to show that $\sum_{P \in \mathcal{P}_G(A,B)} \alpha(z_P)(1 - \sum_{v \in V(P)} x_v) \geq \frac{h(h-1)}{2}t$. To this end, note that $\mathcal{F} \subseteq \mathcal{P}_G(A,B)$. Therefore,

$$\sum_{P \in \mathcal{P}_G(A,B)} \alpha(z_P)(1 - \sum_{v \in V(P)} x_v) = \sum_{F \in \mathcal{F}} f(F)(1 - \sum_{v \in V(F)} x_v).$$

Now, because f is $1/2$-cheap, it holds that $\sum_{v \in V(F)} x_v \leq 1/2$ for any $F \in \mathcal{F}$. Therefore,

$$\sum_{F \in \mathcal{F}} f(F)(1 - \sum_{v \in V(F)} x_v) \leq \frac{1}{2} \sum_{F \in \mathcal{F}} f(F).$$

Recall that $(\mathcal{F},f)$ has power larger than $h(h-1)t$, and hence $\sum_{F \in \mathcal{F}} f(F) \geq h(h-1)t$. Thus, we conclude that indeed $\sum_{P \in \mathcal{P}_G(A,B)} \alpha(z_P)(1 - \sum_{v \in V(P)} x_v) \geq \frac{h(h-1)}{2}t$. □

Let us now conclude our translation.

Lemma 6.21. *Let H be a graph with $h = |V(H)|$. There exist fixed constants $c = c(H)$ and $d = d(H)$ such that given any triple (G, w, t) where G is H-minor free, and given any feasible fractional solution α of* **Pairwise-Flow Hitting LP**(G, w, h, t)*, the following claim holds. Define $\alpha' : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ by $\alpha'(x_v) = d \cdot \alpha(x_v)$ for all $v \in V(G)$. Then, there exists a feasible fractional solution $\alpha^\star$ of* **Well-Linkedness LP**(G, w, ct) *that extends α' and such that $\texttt{cost}(\alpha^\star) = d \cdot \texttt{cost}(\alpha)$.*

Proof. Let $\ell = \ell(H)$ be the constant given in Lemma 6.19. Let $c = 2h(h-1)(\frac{10^4}{10^4\ell^{11}})^{\frac{10^4}{\ell^{10}}+2}$ and $d = 2(\frac{10^4}{\ell^{10}} + 4)$. Suppose, by way of contradiction, that there does not exist a feasible fractional solution $\alpha^\star$ of **Well-Linkedness LP**(G, w, ct) that extends α'.

Note that $ct \geq 1/\ell^5$. Therefore by Lemma 6.19 with respect to α', there exists $\overline{X} \in \texttt{ConPart}(G, h)$ such that for all distinct $i, j \in \{1, 2, \ldots, h\}$, there exists an $(\overline{X}[i], \overline{X}[j])$-flow $(\mathcal{F}_{i,j}, f_{i,j})$ in $G - (X \setminus (\overline{X}[i] \cup \overline{X}[j]))$ (where $X = \bigcup_{\ell=1}^{h} \overline{X}[\ell]$) of power at least $(\frac{\ell^{11}}{10^4})^{\frac{10^4}{\ell^{10}}+2} \cdot ct >$ $h(h-1)t$ and which is $\frac{10^4}{\ell^{10}} + 4$-cheap with respect to α'. By the definition of α', we have that, for all distinct $i, j \in \{1, 2, \ldots, h\}$, $(\mathcal{F}_{i,j}, f_{i,j})$ is $1/2$-cheap with respect to $\alpha|_{\{x_v : v \in V(G)\}}$. Therefore by Lemma 6.20, for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the optimum of **Penalized Flow Packing LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), \alpha, \overline{X}[i], \overline{X}[j])$ is larger than $\frac{h(h-1)}{2}t$. Then, by Observation 5.1, for all $i, j \in \{1, 2, \ldots, h\}$ with $i < j$, the optimum of **Penalized Flow Hitting LP**$(G - (X \setminus (\overline{X}[i] \cup \overline{X}[j])), \alpha, \overline{X}[i], \overline{X}[j])$ is larger than $\frac{h(h-1)}{2}t$. In turn, by Lemma 5.5, this means that $\alpha|_{\{x_v : v \in V(G)\}}$ cannot be extended to a feasible fractional solution of **Pairwise-Flow Hitting LP**(G, w, h, t). However, this is a contradiction as α is an extension of $\alpha|_{\{x_v : v \in V(G)\}}$ that is a feasible fractional solution of **Pairwise-Flow Hitting LP**(G, w, h, t).

So far, we derived that there exists a solution $\alpha^\star$ of **Well-Linkedness LP**(G, w, h, ct) that extends α'. Observe that

$$\texttt{cost}(\alpha^\star) = \texttt{cost}(\alpha') = \sum_{v \in V(G)} w_v \alpha'(x_v) = d \cdot \sum_{v \in V(G)} w_v \alpha(x_v) = d \cdot \texttt{cost}(\alpha).$$

This completes the proof. □

7 Rounding the Well-Linkedness LP

In this section we prove the following theorem.

Theorem 2.1. *There is a polynomial time algorithm that, for any graph H, takes as input an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$, and an integer t, and outputs a vertex set S such that $\texttt{tw}(G) \leq \mathcal{O}(t)$ and $w(S)$ is at most $\mathcal{O}(\log n)$ times the optimum of* **Well-Linkedness LP**(G, w, t)*. That is, $w(S) \leq \mathcal{O}(\log n \cdot \texttt{opt}(G, w, t))$.*

Notice that Theorem 2.1 establishes that the integrality gap of **Well-Linkedness LP**(G, w, t) is at most $\mathcal{O}(\log n)$, at least in the “bicriteria” sense where we are willing to round a fractional solution of **Well-Linkedness LP**(G, w, t) to an integral solution of **Well-Linkedness LP**$(G, w, \mathcal{O}(t))$.

Another important aspect of Theorem 2.1 that is algorithmic—given the graph as input it produces the rounded solution (an assignment to the x-variables) in polynomial time. This is non-trivial because **Well-Linkedness LP**$(G, w, \mathcal{O}(t))$ has an exponential number of both variables and constraints, and so there is no time to actually solve the LP in the process of constructing an integral solution to it. We overcome this problem in the same way as Bansal et al. [7], by appealing to the “round or separate” framework (based on Proposition 4.2).

Bounded distance decompositions. A crucial ingredient of our rounding procedure is the (by now) classic technique of region-decompositions, which have found numerous applications for "cut"-like problems, see e.g. [12, 27, 36, 41].

Let G be a graph, $w : V(G) \to \mathbb{Q}^+$ be a weight function on $V(G)$, and $\alpha : V(G) \to \mathbb{Q}_0^+$ be an assignment of non-negative values to the vertices of G. We define $\texttt{cost}(\alpha) = \sum_{v \in V(G)} w(v)\alpha(v)$. This reflects that α will typically be an assignment to the variables of an LP whose objective function is to minimize the sum of these variables (weighted by w).

Let $d : V(G) \times V(G) \to \mathbb{Q}_0^+$ be the shortest path metric of G with vertex "lengths" given by α. In other words, $d(u, v)$ is equal to the minimum over all $u - v$ paths P of $\sum_{p \in V(P)} \alpha(p)$.

The *weak diameter* of a set $X \subseteq V(G)$ is $\max_{u,v \in X} d(u, v)$. Notice that here we are taking the distances in G, so the shortest paths between u and v may use vertices outside of X. In particular, the weak diameter of S is not necessarily the same as the diamter of $G[X]$.

Let γ be a positive real, a *distance-γ-decomposition* of (G, w, α) is a set $S \subseteq V(G)$ together with a partition of $V(G) \setminus S$ into $X_1, \ldots, X_\ell$ such that every X_i has weak diameter at most γ and satisfies $N(X_i) \subseteq S$. The *cost* of a distance-γ-decomposition is $\sum_{v \in S} w(v)$. We will rely on the following result of Klein, Plotkin and Rao [45].

Proposition 7.1 ([45]). *There is a polynomial time algorithm that, for any graph H, takes as input an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$, an assignment $\alpha : V(G) \to \mathbb{Q}_0^+$, and a positive rational $\gamma \in \mathbb{Q}^+$, and outputs a distance-γ-decomposition $(S, X_1, \ldots, X_\ell)$ of G of cost at most* $\texttt{cost}(\alpha) \cdot \mathcal{O}(\frac{|V(H)|^3}{\gamma})$.

We remark that in Proposition 7.1 as stated in [45], the weights w and assignment α are to the edges of G rather than the vertices, and the set S is a set of edges rather than a vertex set. Further, the result is stated without the weight function w and the "length" assignment α, it is just said that the result may be generalized to weights and lengths by appropriately subdividing and copying edges.

The proofs of Klein, Plotkin and Rao [45] and also Fakcharoenphol and Talwar [28] of the unweighted edge version of Proposition 7.1 carry over without modification to prove the *unweighted* vertex version of Proposition 7.1. However, to prove the *weighted* vertex version (as stated in this paper) we may not just copy vertices and then apply the un-weighted version of the proposition to the unweighted graph with multiple copies of each vertex. The reason for this is that copying vertices may destroy the property of G being H-minor free.

Nevertheless, the proofs of Klein Plotkin and Rao [45] and Fakcharoenphol Talwar [28] of the unweighted edge version of Proposition 7.1 carry over with slight modification to prove Proposition 7.1. All that is required is to replace the BFS-layering of G with a layering of G into layers of width $\mathcal{O}(\frac{\gamma}{|V(H)|^3})$ in the metric d, and replacing the deletion of a BFS-layer with an application of max-flow/min-cut to the layer in the metric that is being deleted.

7.1 Balanced Separators

Throughout this section, let H be a graph, G be an H-minor free graph, $w : V(G) \to \mathbb{Q}^+$ be a weight function on the vertices, and t be an integer. Let $\alpha : \{x_v : v \in V(G)\} \to \mathbb{Q}_0^+$ be an assignment. For the sake of brevity, we sometimes treat α as if it is a function from $V(G)$ to $\mathbb{Q}_0^+$ (where $\alpha(v) = \alpha(x_v)$ for every vertex $v \in V(G)$). We say that a vertex v has *large weight* if $w(v) \cdot \frac{t}{\texttt{cost}(\alpha)} \geq 1$.

Lemma 7.1. *There exists a polynomial time algorithm that given G, w, t, α, and a subset $S \subseteq V(G)$, finds either*

- *a separating hyperplane witnessing that α is infeasible for* **Well-Linkedness LP**(G, w, t)*, or*

- *a subset $X \subseteq V(G)$ of $\mathcal{O}(t|V(H)|^3)$ vertices of large weight, and a subset $Y \subseteq V(G)$ of weight $w(Y) \leq \mathcal{O}(\mathtt{cost}(\alpha)|V(H)|^3)$ that contains no vertices of large weight, such that every connected component of $G - (X \cup Y)$ contains no more than $\frac{4}{5}|S|$ vertices of S.*

Proof. The algorithm applies Proposition 4.2 with respect to **Well-Linkedness LP**(G, w, t) and the subset $S \subseteq V(G)$, and finds either a separating hyperplane witnessing that α is infeasible for **Well-Linkedness LP**(G, w, t) or a feasible assignment $\beta^S : \{y_v : v \in V(G)\} \cup \{d_{uv} : u, v \in V(G)\} \to \mathbb{Q}^+$ to **sep-LP**(G, S, α) of cost at most t. In the first case, we are done. Now, we proceed with the second case.

Let $\alpha'(v) = \alpha(v) + \beta^S(v)$ for every $v \in V(G)$, and let $d'(uv) = \beta^S(d_{uv})$ be the shortest path metric of G with respect to the vertex lengths given by $\alpha'(v)$. Observe that since β^S is a feasible solution to **sep-LP**(G, S, α), it follows that

$$\sum_{v \in U} d'(uv) \geq |U| - \frac{|S|}{2}$$

for every choice of $U \subseteq S$ and $u \in U$.

Furthermore, let $w'(v) = \min(1, w(v) \cdot \frac{t}{\mathtt{cost}(\alpha)})$. Note that $w'(v) = 1$ if and only if v has large weight. Additionally, notice that

$$\sum_{v \in V(G)} w'(v)\alpha'(v) \leq \sum_{v \in V(G)} \left(w(v) \cdot \frac{t}{\mathtt{cost}(\alpha)} \cdot \alpha(v) + \beta^S(v) \right) \leq 2t.$$

We apply the algorithm of Proposition 7.1 to G with weight function w', assignment α', and $\gamma = \frac{1}{10}$. This algorithm outputs a bounded distance decomposition $(Z, C_1, C_2, \ldots, C_r)$ of G such that

$$\sum_{v \in Z} w'(v) = \sum_{v \in V(G)} w'(v)\alpha'(v)\mathcal{O}(|V(H)|^3) \leq \mathcal{O}(t \cdot |V(H)|^3).$$

Further, the weak diameter of each C_i with respect to d' is at most $1/10$.

We claim that for every C_i, $|C_i \cap S| \leq \frac{4}{5}|S|$. Suppose not, and let $U = C_i \cap S$ and $u \in U$. We have that

$$\frac{|U|}{10} \geq \sum_{v \in U} d'(uv) \geq |U| - \frac{|S|}{2} \geq \frac{4}{5}|S| - \frac{1}{2}|S|,$$

which is a contradiction.

We partition Z into X and Y as follows $X = \{v \in Z : w'(Z) = 1\}$ and $Y = Z \setminus X$. Since a vertex has large weight if and only if $w'(v) = 1$, it follows that X contains only vertices of large weight, and Y contains only vertices that do not have large weight. Further, since $\sum_{v \in Z} w'(v) = \mathcal{O}(t \cdot |V(H)|^3)$, it follows that $|X| = \mathcal{O}(t \cdot |V(H)|^3)$ and

$$w(Y) \leq \mathcal{O}(t \cdot |V(H)|^3) \cdot \frac{\mathtt{cost}(\alpha)}{t} \leq \mathcal{O}(\mathtt{cost}(\alpha) \cdot |V(H)|^3).$$

This concludes the proof. ◻

We will apply Lemma 7.1 in a recursive fashion, following Robertson and Seymour's approximation algorithm for treewidth [53], or rather the way it is adapted by Bansal et al. [7]. We will need our recursion tree to have logarithmic depth, thus we will need to evenly split *both* the entire vertex set of the graph *and* the current "root bag" of the decomposition.

We will need the following simple observation.

Observation 7.1. *Let $Z = \{z_1, \ldots, z_\ell\}$ be a set of positive reals such that $\sum_i z_i = 1$ and $z_i \leq \frac{4}{5}$. Then, there exists a partition of Z into two sets, Z_1 and Z_2, where $\sum_{z_j \in Z_i} z_j \leq \frac{9}{10}$ for each $i \in \{1, 2\}$.*

Proof. Initially, set $Z_1 = Z_2 = \emptyset$. Then, for every $z_i \in Z$, add it to Z_1 unless doing so would increase $\sum_{z_j \in Z_1} z_j$ above $\frac{9}{10}$. In that case, add z_i to Z_2. Clearly, Z_1 and Z_2 form a partition of Z. Further, $Z_1 \leq \frac{9}{10}$ by construction. Hence Z_2 is not empty. Let $z_i \in Z_2$, and observe that when z_i was inserted into Z_2, inserting z_i into Z_1 would have made $\sum_{z_j \in Z_1} z_j$ bigger than $\frac{9}{10}$. Since $z_i \leq \frac{8}{10}$, it follows that $\sum_{z_j \in Z_1} z_j > \frac{1}{10}$, which implies $\sum_{z_j \in Z_2} z_j \leq \frac{9}{10}$. $\square$

Lemma 7.2. *For every graph H, there exists a constant $c = \mathcal{O}(|V(H)|^3)$ and a polynomial time algorithm that given an H-minor free graph G, w, t, α, and a subset $\widehat{S} \subseteq V(G)$, outputs either*

- *a separating hyperplane witnessing that α is infeasible for* **Well-Linkedness LP**(G, w, t)*, or*
- *a subset $X \subseteq V(G)$ of at most ct vertices of large weight together with a subset $Y \subseteq V(G)$ such that $w(Y) \leq c \cdot \texttt{cost}(\alpha)$ and Y contains no vertices of large weight, and a partition of $V(G) \setminus (X \cup Y)$ into four sets Z_1, Z_2, Z_3, Z_4 such that for every i, $N(Z_i) \subseteq (X \cup Y)$, $|Z_i| \leq \frac{9}{10}n$, and $|Z_i \cap \widehat{S}| \leq \frac{9}{10}|\widehat{S}|$.*

Proof. The algorithm first applies Lemma 7.1 with $S = V(G)$, and obtains either a separating hyperplane witnessing that α is infeasible for **Well-Linkedness LP**(G, w, t) (which the algorithm then outputs and terminates), or a set X_V of $\mathcal{O}(t|V(H)|^3)$ vertices of large weight, and a set Y_V such that $w(Y_V) \leq \mathcal{O}(\texttt{cost}(\alpha)|V(H)|^3)$, Y_V contains no vertices of large weight, and every connected component of $G - (X_V \cup Y_V)$ contains no more than $\frac{4}{5}n$ vertices of $V(G)$.

Apply Observation 7.1 to the (sizes of) the connected components of $G - (X_V \cup Y_V)$ (relative to the total number of vertices in them) to obtain a partition of the connected components of $G - (X_V \cup Y_V)$ in two sets, Z_L and Z_R, each of size at most $\frac{9}{10}n$.

Now, apply Lemma 7.1 with $S = \widehat{S}$, and obtain either a separating hyperplane witnessing that α is infeasible for **Well-Linkedness LP**(G, w, t) (which the algorithm then outputs and terminates), or a set X_S of $\mathcal{O}(t|V(H)|^3)$ vertices of large weight, and a set Y_S such that $w(Y_S) \leq \mathcal{O}(\texttt{cost}(\alpha)|V(H)|^3)$, Y_S contains no vertices of large weight, and every connected component of $G - (X_S \cup Y_S)$ contains no more than $\frac{4}{5}|\widehat{S}|$ vertices of $\widehat{S}$.

We set $X = X_V \cup X_S$ and $Y = Y_V \cup Y_S$, and observe that $|X| = \mathcal{O}(|V(H)|^3 t)$ and $w(Y) = \mathcal{O}(|V(H)|^3 \cdot \texttt{cost}(\alpha))$. Further, X only contains vertices of large weight and Y only contains vertices of small weight.

Consider the connected components $C_1, \ldots, C_\ell$ of $G[Z_L] - (X_S \cup Y_S)$. Applying Observation 7.1 to $|C_i \cap \widehat{S}|/|\widehat{S} \cap (Z_L \setminus (X_S \cup Y_S))|$ we can partition the components C_i into two sets Z_1 and Z_2 such that for every $i \in \{1, 2\}$, $|Z_i \cap \widehat{S}| \leq \frac{9}{10}|\widehat{S}|$. Similarly, partition the connected components of $G[Z_R] - (X_S \cup Y_S)$ and into two sets Z_3 and Z_4 such that for every $i \in \{3, 4\}$, $|Z_i \cap \widehat{S}| \leq \frac{9}{10}|\widehat{S}|$. Since each Z_i is a subset of either Z_L or Z_R, $|Z_i| \leq \frac{9}{10}n$, concluding the proof. $\square$

7.2 Rounding Algorithm

Before we turn to describe the main rounding algorithm, we may delete all vertices that have been "picked" to extent at least $\frac{1}{\log n}$, that is, all vertices $v \in V(G)$ with $\alpha(x_v) \geq \frac{1}{\log n}$. The next lemma essentially proves Theorem 2.1 under the assumption that there are no vertices picked to extent at least $\frac{1}{\log n}$.

Lemma 7.3. *For any graph H, there exists a constant c and a polynomial time algorithm that takes as input an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$, an integer t, a vertex set $\widehat{S} \subseteq V(G)$ of size at most ct, and an assignment $\alpha : V(G) \to \mathbb{Q}_0^+$ such that $\alpha(v) \leq \frac{1}{\log n}$ for every $v \in V(G)$, and outputs either*

- *a separating hyperplane witnessing that α is infeasible for* **Well-Linkedness LP**(G, w, t)*, or*

- *a vertex set Y and a tree-decomposition of $G - Y$ of width at most $2ct$ such that $\widehat{S}$ is the root bag; furthermore, $w(Y) \leq \mathcal{O}(\log n \cdot \mathtt{cost}(\alpha))$ and Y contains no vertices of large weight.*

Proof. **Weight reduction.** The requirement that Y should contain no vertices of large weight will allow us to work with a modified weight function where all vertices of large weight have weight upper bounded by $\frac{\mathtt{cost}(\alpha)}{t}$. More precisely, we define the weight function $w^\star : V(G) \to \mathbb{Q}^+$ as follows. For every vertex $v \in V(G)$, let $w^\star(v) = \min\{w(v), \frac{\mathtt{cost}(\alpha)}{t}\}$. Clearly, modification of weights does not affect the feasibility of a solution. We denote the cost of α with respect to $w^\star$ by $\mathtt{cost}^\star$. The definition of vertices with large weight does not change, that is, vertices of large weight remains those of weight at least $\frac{\mathtt{cost}(\alpha)}{t}$. The new weight function $w^\star$ will only be useful when we will make recursive calls.

Basis. Our algorithm is a recursive algorithm that works as follows. In the basis, if G already has at most $2ct$ vertices, then the algorithm returns $Y = \emptyset$ and a tree decomposition that contains two bags, the root bag $\widehat{S}$ with one child whose bag is $V(G)$.

Step. Now, suppose that G has more than $2ct$ vertices. Then, the algorithm applies Lemma 7.2 to $G, w, t, \alpha, \widehat{S}$ and obtains either *(i)* a separating hyperplane witnessing that α is infeasible for **Well-Linkedness LP**(G, w, t), or *(ii)* a set X of at most $\mathcal{O}(t)$ vertices of large weight together with a set $\widehat{Y}$ such that $w(\widehat{Y}) \leq \mathcal{O}(\mathtt{cost}(\alpha))$ and $\widehat{Y}$ contains no vertices of large weight, and a partition of $V(G) \setminus (X \cup \widehat{Y})$ into four sets Z_1, Z_2, Z_3, Z_4 such that for every i, $N(Z_i) \subseteq (X \cup \widehat{Y})$, $|Z_i| \leq \frac{9}{10}n$, and $|Z_i \cap \widehat{S}| \leq \frac{9}{10}|\widehat{S}|$. In the first case, the algorithm outputs this hyperplane and terminates, while in the second case, it proceeds as follows.

The algorithm calls itself recursively on $G[Z_i \cup X]$ for every $i \in \{1, 2, 3, 4\}$, with the "top bag" set $\widehat{S}_i$ of the recursive call on $G[Z_i \cup X]$ being $\widehat{S}_i = X \cup (\widehat{S} \cap Z_i)$. The weight function w_i and assignment α_i to $Z_i \cup X$ in the i-th recursive call are the restrictions of $w^\star$ and α, respectively, to $Z_i \cup X$. Notice that here, we make use of $w^\star$ rather than w, which will be essential for correctness later. We observe that $|\widehat{S}_i| \leq |X| + \frac{9}{10}|\widehat{S}| \leq ct$, assuming that the constant c is large enough compared to the constant hidden in the $\mathcal{O}(t)$ upper bound on $|X|$ in Lemma 7.2. Furthermore, $\alpha(v) \leq \frac{1}{\log n} \leq \frac{1}{\log |Z_i \cup S|}$, and hence the premises for running the algorithm recursively on $G[Z_i \cup X]$ are satisfied.

If at least one of the four recursive calls returns a separating hyperplane witnessing that α_i is infeasible for **Well-Linkedness LP**$(G[Z_i \cup X], w_i, t)$, then the same hyperplane witnesses that α is infeasible for **Well-Linkedness LP**$(G, w^\star, t)$ (and hence for **Well-Linkedness LP**(G, w, t)). Indeed, this statement holds because deleting vertices from G and restricting a feasible solution to **Well-Linkedness LP**$(G, w^\star, t)$ accordingly cannot turn the feasible solution into an infeasible solution (though it can turn an infeasible solution to a feasible one). In this case, the algorithm outputs this hyperplane and terminates.

Suppose now that each of the four recursive calls returns with a set Y_i, and a tree-decomposition of $G[(Z_i \cup X)] - Y_i$ of width at most $2ct$, with $X \cup (\widehat{S} \cap Z_i)$ as the top bag. We then set $Y = \widehat{Y} \cup \bigcup_{i=1}^4 Y_i$, and construct a tree-decomposition of $G - Y$ by making a root bag $\widehat{S}$, with one child $\widehat{S} \cup X$. This child, in turn, has four children—the roots of the four tree decompositions of $G[(Z_i \cup X)] - Y_i$, $i \in \{1, 2, 3, 4\}$. Recall that the root bag of each of these decompositions is $X \cup (\widehat{S} \cap Z_i)$. Thus, it is easy to verify that the constructed decomposition is indeed a tree decomposition of $G - Y$. Further, $|\widehat{S}| \leq ct$ and $|X| \leq \frac{c}{10}t$, and hence both of the new bags of the tree decomposition have size at most $2ct$. Hence the width of this tree decomposition is at most $2ct$, as claimed.

Correctness. The algorithm clearly terminates in polynomial time. Moreover, recall that $Y = \widehat{Y} \cup \bigcup_{i=1}^4 Y_i$, and that $\widehat{Y}$ does not have vertices of large weight. Moreover, the weight of every vertex in Y_i, $i \in \{1,2,3,4\}$, is not large with respect to w_i, and hence it is upper bounded by $\frac{\texttt{cost}^\star(\alpha_i)}{t} \leq \frac{\texttt{cost}^\star(\alpha)}{t} \leq \frac{\texttt{cost}(\alpha)}{t}$. Therefore, the set Y does not contain vertices of large weight.

Hence, all that remains to prove is that $w(Y) = \mathcal{O}(\log n \cdot \texttt{cost}(\alpha))$. To this end, we let $\zeta(\widehat{\alpha}, n)$ be the largest weight of a set Y output by the algorithm when run on a graph with n vertices and an assignment α with cost $\widehat{\alpha}$. We would like to prove that there exists a constant ρ such that $\zeta(\widehat{\alpha}, n) \leq \rho \cdot \log n \cdot \widehat{\alpha}$. For this purpose, we further let $\zeta_d(\ell, \widehat{\alpha}, n)$ be the largest weight of a set Y output by a recursive call to the algorithm that is made at depth d in the recursion tree, where the recursive call is made on a graph with $\ell \leq n$ vertices and an assignment α with cost $\widehat{\alpha}$, and the initial call is made on a graph with n vertices. We prove by induction on ℓ that there exists a constant ρ such that

$$\zeta_d(\ell, \widehat{\alpha}, n) \leq (\frac{\log n - 4c}{\log n})^d \cdot \rho \cdot \log n \cdot \widehat{\alpha}.$$

In the initial call, $d = 0$ and $\ell = n$, and thus it would follows that $\zeta(\widehat{\alpha}, n) \leq \rho \cdot \log n \cdot \widehat{\alpha}$.

In the base case, where $\ell \leq 2ct$, $Y = \emptyset$ and the inequality follows for any d and n.

For the inductive step, consider an integer $\ell > 2ct$, a value $\widehat{\alpha}$ and and integer n, and consider the instance $G, \widehat{S}, w, \alpha, t$ on ℓ vertices with $\texttt{cost}(\alpha) = \widehat{\alpha}$ that maximizes $w(Y)$ for the output set Y. Assume now that the statement has been proved for all graphs on less than ℓ vertices.

The algorithm will perform four recursive calls to $G[Z_i \cup X]$ and output $Y = \widehat{Y} \cup (\bigcup_{i=1}^{4} Y_i)$. By Lemma 7.2, we have that $w(\widehat{Y}) \leq c' \cdot \widehat{\alpha}$ for a constant c' that depends only on H. Let α_i be the restriction of α to $G[Z_i \cup X]$. From the inductive hypothesis and because Y does not contain vertices of large weight, we have that $w(Y_i) = w^\star(Y_i) \leq (\frac{\log n - 4c}{\log n})^{d+1} \cdot \rho \cdot \log n \cdot \texttt{cost}^\star(\alpha_i)$. Hence, we have that

$$w(Y) \leq w(\widehat{Y}) + \sum_{i=1}^{4} w(Y_i) \leq c' \cdot \widehat{\alpha} + (\frac{\log n - 4c}{\log n})^{d+1} \cdot \rho \cdot \log n \cdot \sum_{i=1}^{4} \texttt{cost}^\star(\alpha_i).$$

Now, observe that $\sum_{i=1}^{4} \texttt{cost}^\star(\alpha_i) \leq \widehat{\alpha} + 3 \sum_{v \in X} w^\star(v)\alpha(v)$, because the vertices of X are the only ones that appear in more than one of the instances. Furthermore, $w^\star(v) \leq \frac{\texttt{cost}(\alpha)}{t}$ for all vertices $v \in V(G)$. Since $\alpha(v) \leq \frac{1}{\log n}$ for every vertex $v \in V(G)$, we have that $3 \sum_{v \in X} w^\star(v)\alpha(v) \leq 3|X| \cdot \frac{\widehat{\alpha}}{t \log n} \leq \frac{3c\widehat{\alpha}}{\log n}$. In the latter inequality, we used the fact that $|X| \leq ct$. Hence, we have that

$$\begin{aligned}
w(Y) &\leq c' \cdot \widehat{\alpha} + (\frac{\log n - 4c}{\log n})^{d+1} \cdot \rho \cdot \log n \cdot \sum_{i=1}^{4} \texttt{cost}^\star(\alpha_i) \\
&\leq c' \cdot \widehat{\alpha} + (\frac{\log n - 4c}{\log n})^{d+1} \cdot \rho \cdot \log n \cdot (\widehat{\alpha} + 3 \sum_{v \in X} w^\star(v)\alpha(v)) \\
&\leq c' \cdot \widehat{\alpha} + (\frac{\log n - 4c}{\log n})^{d+1} \cdot \rho \cdot \log n \cdot (\widehat{\alpha} + \frac{3c\widehat{\alpha}}{\log n}) \\
&\leq (\frac{\log n - 4c}{\log n})^{d} \cdot \Big((\frac{\log n}{\log n - 4c})^{d} \cdot c' \cdot \widehat{\alpha} + \frac{\log n - 4c}{\log n} \cdot \rho \cdot \log n \cdot (\widehat{\alpha} + \frac{3c\widehat{\alpha}}{\log n}) \Big).
\end{aligned}$$

Notice that the depth of the recursion is upper bounded by $\widetilde{c} \cdot \log n$ for some constant $\widetilde{c}$ independent of the input (because each recursive call decreases the number of vertices by a constant fraction of n, namely, $\frac{1}{10}n$). Thus, $(\frac{\log n}{\log n - 4c})^d \leq (\frac{\log n}{\log n - 4c})^{\widetilde{c} \log n} = 1/(1 - \frac{4c}{\log n})^{\widetilde{c} \log n} \leq e^{4c\widetilde{c}}$. Denote $\widehat{c} = e^{4c\widetilde{c}} \cdot c'$. Then,

$$w(Y) \leq (\frac{\log n - 4c}{\log n})^{d} \cdot \Big(\widehat{c} \cdot \widehat{\alpha} + \rho \cdot (\log n - 4c) \cdot (\widehat{\alpha} + \frac{3c\widehat{\alpha}}{\log n}) \Big).$$

Thus, to prove that $w(Y) \leq (\frac{\log n - 4c}{\log n})^d \cdot \rho \cdot \log n \cdot \widehat{\alpha}$, it suffices to assert that

$$\widehat{c} \cdot \widehat{\alpha} + \rho \cdot (\log n - 4c) \cdot (\widehat{\alpha} + \frac{3c\widehat{\alpha}}{\log n}) \leq \rho \cdot \log n \cdot \widehat{\alpha}.$$

To this end, note that

$$\widehat{c} \cdot \widehat{\alpha} + \rho \cdot (\log n - 4c) \cdot (\widehat{\alpha} + \frac{3c\widehat{\alpha}}{\log n}) \quad \leq \rho \cdot \log n \cdot \widehat{\alpha} + \widehat{c} \cdot \widehat{\alpha} - \rho \cdot \widehat{\alpha}.$$

Thus, the proof is complete by selecting $\rho = \widehat{c}$. $\square$

As we have already mentioned above, the correctness of Theorem 2.1 readily follows from Lemma 7.3. To see this, consider an execution of the ellipsoid algorithm where instead of the standard use of a separation oracle, we call the algorithm in Lemma 7.3 with $\widehat{S} = \emptyset$. If the algorithm returns a subset $Y \subseteq V(G)$, then $\mathtt{tw}(G - Y) \leq \mathcal{O}(t)$ and $w(S) \leq \mathcal{O}(\log n \cdot \mathtt{opt}(G, w, t))$, and hence we are done. Else, we resume the execution of the ellipsoid algorithm with the separating hyperplane. As the number of times separating hyperplanes can be discovered is polynomial (by the correctness of the ellipsoid algorithm), the theorem follows.

8 Proof of the Scaling Lemma and its Extensions

We are now ready to prove Lemma 1.1, we re-state it here for convenience.

Lemma 1.1. (Scaling Lemma) *There exists an algorithm that given an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}^+$, and positive integers t and s, in polynomial time outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot \mathtt{opt}(G, w, t)/s$ such that $\mathtt{tw}(G - S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on H, $\mathtt{tw}(G - S)$ is the treewidth of $G - S$, and $\mathtt{opt}(G, w, t)$ is the minimum weight of a subset $U \subseteq V(G)$ such that $\mathtt{tw}(G - U) \leq t$.*

Proof. Given an input (G, w, t), we describe a polynomial time algorithm with the claimed properties. Let c_1, c_2, d_1, d_2 be some fixed constants that only depend on H, which are implicitly determined by the analysis below. Towards the description of the algorithm, consider the following hypothetical steps (that the algorithm does *not* execute).

Step A. Let α be an optimal fractional solution of **Well-Linkedness LP**(G, w, t).

Step B. Translate α into a fractional solution α_1 of **Grid Hitting LP**$(G, w, c_1 t)$ such that $\mathtt{cost}(\alpha_1) = (1/t) \cdot \mathtt{cost}(\alpha)$. (Lemma 2.2.)

Step C. Then, scale the feasible solution α_1 of **Grid Hitting LP**$(G, w, c_1 t)$ to get a feasible fractional solution α_2 of **Grid Hitting LP**$(G, w, s \cdot c_1 t)$ such that $\mathtt{cost}(\alpha_2) = \mathtt{cost}(\alpha_1)/s^2$. (Lemma 2.1.)

Step D. Now, given a feasible fractional solution α_2 of **Grid Hitting LP**$(G, w, s \cdot c_1 t)$ such that $\mathtt{cost}(\alpha_2) = \mathtt{cost}(\alpha_1)/s^2$, we obtain a feasible fractional solution $\alpha^\star$ of **Well-Linkedness LP**$(G, w, s \cdot c_2 c_1 t)$ of cost $d_1 st \cdot \mathtt{cost}(\alpha_2)$. (Lemmas 2.4 and 2.5.)

Step E. Finally, apply Theorem 2.1 to obtain a vertex set S such that $\mathtt{tw}(G - S) \leq c \cdot st$ and $w(S)$ is at most $\mathcal{O}(\log n)$ times the optimum of **Well-Linkedness LP**$(G, w, s \cdot c_2 c_1 t)$. As the optimum of **Well-Linkedness LP**$(G, w, s \cdot c_2 c_1 t)$ is upper bounded by $\mathtt{cost}(\alpha^\star)$, we have that

$$\begin{aligned}
w(S) &\leq d_2 \log n \cdot \mathtt{cost}(\alpha^\star) \\
&\leq d_2 \log n \cdot d_1 st \cdot \mathtt{cost}(\alpha_2) \\
&\leq d_2 \log n \cdot d_1 st \cdot \frac{\mathtt{cost}(\alpha_1)}{s^2} \\
&\leq d_2 \log n \cdot d_1 st \cdot \frac{\mathtt{cost}(\alpha)}{s^2 t} \\
&= d_2 \log n \cdot d_1 \cdot \frac{\mathtt{cost}(\alpha)}{s} \\
&\leq d_2 \log n \cdot d_1 \cdot \frac{\mathtt{opt}(G, w, t)}{s}.
\end{aligned}$$

Our algorithm only executes the last step. That is, it calls the algorithm in Theorem 2.1 to obtain a vertex set S such that $\mathtt{tw}(G-S) \leq c \cdot st$ and $w(S)$ is at most $\mathcal{O}(\log n)$ times the optimum of **Well-Linkedness LP**$(G, w, s \cdot c_2 c_1 t)$. By the analysis above, $w(S) \leq d \log n \cdot \mathtt{opt}(G, w, t)/s$ for a fixed constant d. Note that c and d depend only on H, and since the algorithm in Theorem 2.1 runs in polynomial time, so does our algorithm. This concludes the proof. □

Given Lemma 1.1 we now prove Lemma 1.3.

Lemma 1.3. *There exists an algorithm that given an H-minor free graph G, a weight function $w : E(G) \to \mathbb{Q}^+$, and positive integers t and s, in polynomial time outputs a subset $S \subseteq E(G)$ of weight at most $d \log n \cdot \mathtt{opt}_E(G, w, t)/s$ such that $\mathtt{tw}(G - S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on H, and $\mathtt{opt}_E(G, w, t)$ is the minimum weight of a subset $U \subseteq E(G)$ such that $\mathtt{tw}(G - U) \leq t$.*

Proof. It is sufficient to transform the edge weighted input instance (G, w) to a vertex weighted instance (G', w') where G' is equal to G with all edges subdivided, and w' assigns weight $2\sum_{e \in E(G)} w(e)$ (which essentially plays the role of ∞) to vertices of G' that correspond to vertices of G, and for every vertex $x_e \in V(G')$ that corresponds to an edge $e \in E(G)$ we set $w'(x_e) = w(e)$.

It is well-known that the size of the largest clique minor of a graph G can not grow when edges of G are subdivided, hence G' excludes the complete graph on $|V(H)|$ vertices as a minor. We apply Lemma 1.1 to G', w' and obtain a set $S' \subseteq V(G')$ such that $w'(S') \leq d \log n \cdot \mathtt{opt}(G', w', t)/s$ such that $\mathtt{tw}(G' - S') \leq c \cdot st$. Here c and d are constants that depend only on $|V(H)|$.

Without loss of generality S' does not contain any vertices of G' that are also vertices of G. Otherwise $w(S') \geq 2\sum_{e \in E(G)} w(e)$. On the other the set $S^* = \{x_e : e \in E(G)\}$ satisfies $w(S^*) = \sum_{e \in E(G)} w(e)$ and $\mathtt{tw}(G' - S^*) = 0$ so we can choose $S' = S^*$ instead. Define $S = \{e \in E(G) : x_e \in S'\}$. We claim that S satisfies the statement of the lemma.

First we have that $\mathtt{tw}(G - S) \leq \mathtt{tw}(G' - S') \leq c \cdot st$ because $G - S$ is a minor of $G' - S'$. Second we have that $\mathtt{opt}_E(G, w, t) \leq \mathtt{opt}(G', w', t')$ using the same reasoning. Specifically, let $X' \subseteq E(G')$ be the minimum weight set such that $\mathtt{tw}(G' - X') \leq t$. Without loss of generality $X' \subseteq S^*$, and $X = \{e \in E(G) : x_e \in X'\}$ satisfies $w(X) = w'(X') = \mathtt{opt}(G', w', t')$ and $\mathtt{tw}(G - X) \leq \mathtt{tw}(G' - X') \leq t$. We conclude that $\mathtt{opt}_E(G, w, t) \leq \mathtt{opt}(G', w', t')$, as claimed. By definition of S' and S we have

$$w(S) = w'(S') \leq d \log n \cdot \mathtt{opt}(G', w', t)/s \leq d \log n \cdot \mathtt{opt}(G, w, t)/s$$

This concludes the proof. □

Given Lemma 1.3 we prove Lemma 1.4.

Lemma 1.4. *There exists an algorithm that given a graph G of genus g, a weight function $w : V(E) \to \mathbb{Q}^+$, and positive integers t and s, in polynomial time outputs a subset $S \subseteq E(G)$ of weight at most $d \log n \cdot \mathtt{opt}_{EC}(G, w, t)/s$ such that $\mathtt{tw}(G/S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on g and $\mathtt{opt}_{EC}(G, w, t)$ is the minimum weight of a subset $U \subseteq E(G)$ such that $\mathtt{tw}(G/U) \leq t$.*

Proof. We shall use the following facts about graphs embedded in a surface of genus g.

1. Every graph G embedded in a surface of genus g has a *dual* graph G^* which is also embedded in the same surface.
2. The dual of G^* is G.
3. There is a one-to-one correspondence between the edges of G and the edges of the dual G^*. For every edge set $S \subseteq E(G)$ the dual of G/S is $G^* - S$.
4. $\mathtt{tw}(G) - g - 1 \leq \mathtt{tw}(G^*) \leq \mathtt{tw}(G) + g + 1$.

The first three are standard facts shown in the book of Mohar and Thomassen [51], while the last was proved by Mazoit [50].

We apply Lemma 1.3 to G^* and obtain a set S such that $w(S) \leq d \log n \cdot \mathtt{opt}_E(G^*, w, t + g + 1)/s$ and $\mathtt{tw}(G^* - S) \leq c \cdot s(t + g + 1)$. We claim that S satisfies the statement of the lemma.

We have that $\mathtt{tw}(G - S) \leq \mathtt{tw}(G^* - S) + g + 1 \leq c \cdot s(t + g + 1) + g + 1 \leq c' \cdot st$ where c' is chosen such that the last inequality holds (choosing $c' = c \cdot 3(g + 1)$ suffices). Further, we have that $w(S) \leq d \log n \cdot \mathtt{opt}_E(G^*, w, t + g + 1)/s \leq d \log n \cdot \mathtt{opt}_{EC}(G, w, t)/s$ because every set $X \subseteq E(G)$ that satisfies $\mathtt{tw}(G/X) \leq t$ satisfies $\mathtt{tw}(G^* - X) \leq t + g + 1$ as well. This concludes the proof. $\square$

8.1 Non-Constructive Scaling Lemma for Vertex Contraction

In this subsection we will prove Lemma 1.5. We restate the lemma here for ease of reference.

Lemma 1.5. *There exists an algorithm that given a graph G of genus g, a weight function $w : V(G) \to \mathbb{Q}^+$, positive integers t and s, and a subset $U \subseteq V(G)$ such that $\mathtt{tw}(G/U) \leq t$, in polynomial time outputs a subset $S \subseteq V(G)$ of weight at most $d \log n \cdot w(U)/s$ such that $\mathtt{tw}(G/S) \leq c \cdot st$. Here, d and c are fixed constants that depend only on g.*

For our proof we will first need to prove a property of apex-minor free graphs. Towards this, we need the following equivalent characterization of minors.

Proposition 8.1 ([24])**.** *A graph H is a minor of G if and only if there is a map $\phi : V(H) \to 2^{V(G)}$ such that for every vertex $v \in V(H)$, $G[\phi(v)]$ is connected, for every pair of vertices $v, u \in V(H)$, $\phi(u) \cap \phi(v) = \emptyset$, and for every edge $uv \in E(H)$, there is an edge $u'v' \in E(G)$ such that $u' \in \phi(u)$ and $v' \in \phi(v)$.*

Grids and triangulated grids. Given a positive integer t, we denote by $\boxplus_t$ the $t \times t$ grid. Recall that, for a positive integer t, a $t \times t$ *grid* $\boxplus_t$ is a graph H whose vertex set can be denoted by $\{v_{i,j} : i, j \in \{1, 2, \ldots, t\}\}$ so that $E(H) = \{\{v_{i,j}, v_{i',j'}\} : |i - i'| + |j - j'| = 1\}$. For an integer $t > 0$, the graph Γ_t is obtained from the grid $\boxplus_t$ by adding, for all $1 \leq x, y \leq t - 1$, the edge $\{v_{x+1,y}, v_{x,y+1}\}$, and additionally making vertex $v_{t,t}$ adjacent to all the other vertices $v_{x,y}$ with $x \in \{1, t\}$ or $y \in \{1, t\}$, i.e., to the whole border of $\boxplus_t$. Graph Γ_6 is shown in Fig. 1.

Lemma 8.1 ([30])**.** *For every apex graph H, there is a $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k$ contains Γ_k as a contraction.*

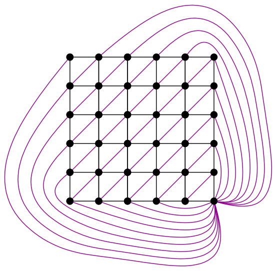

Figure 1: Graph Γ_6.

In the next lemma we show that if G excludes an apex graph as a minor and we *contract* vertex disjoint starts in G to get $G^\star$ then, up to constant factors, the treewidth of G is at most the treewidth of $G^\star$.

Lemma 8.2. *Let H be an apex graph and let G be an H-minor free graph. Furthermore, let $S_1, \ldots, S_\ell$ be the vertex sets of pairwise vertex-disjoint stars in G, and let $G^\star$ be the graph obtained by contracting the edges of $G[S_1], \ldots, G[S_\ell]$. Then, $\mathtt{tw}(G) \leq c_H \cdot (\beta \cdot \mathtt{tw}(G^\star) + 10)$. Here $\beta = 100$.*

Proof Sketch. Let $k = \beta \cdot \mathtt{tw}(G^\star)$ and $\ell = \mathtt{tw}(G^\star)$. For our proof we will demonstrate a $\boxplus_\ell$ in $G^\star$. Towards a contradiction assume that $\mathtt{tw}(G) > c_H \cdot (k + 10)$. Then by Lemma 8.1, we have that G contains Γ_{k+10} as a contraction. We view Γ_{k+10} as follows; $\phi : V(\Gamma_{k+10}) \rightarrow 2^{V(G)}$ such that for every vertex $v \in V(\Gamma_{k+10})$, $G[\phi(v)]$ is connected, for every pair of vertices $v, u \in V(\Gamma_{k+10})$, $\phi(u) \cap \phi(v) = \emptyset$, and for every edge $uv \in E(\Gamma_{k+10})$, there is an edge $u'v' \in E(G)$ such that $u' \in \phi(u)$ and $v' \in \phi(v)$. Let Δ_k denote the induced subgraph of Γ_k obtained by deleting first 5 outer layers of it. That is, we get an insulation of 5 layers around the boundary of Δ_k. Now let the rows of Δ_k be denoted by $R_1, \ldots, R_k$. That is, R_i consists of $\{v_{5+i,j} \mid j \in \{6, \ldots, k+5\}\}$. Similarly, the columns of Δ_k is denoted by $C_1, \ldots, C_k$. That is, C_j consists of $\{v_{i,5+j} \mid i \in \{6, \ldots, k+5\}\}$.

For sake of brevity, we assume that the vertices in Δ_k are named $w_{i,j}$ where $1 \leq i, j \leq k$. Now let $\boxplus_\ell$ be an $\ell \times \ell$ grid. Furthermore, assume that the vertices in $\boxplus_\ell$ are named $x_{i,j}$, where $1 \leq i, j \leq \ell$. We associate $x_{i,j}$ to the vertex $w_{i',j'}$ where $i' = (i-1)\beta + 1$ and $i' = (j-1)\beta + 1$. Now we take a 3×3 grid around $w_{i',j'}$ and denote it by $\mathbb{T}_{i',j'}$. For every vertex $x_{i,j}$ we associate a vertex subset $V_{i,j} = \cup_{v \in V(\mathbb{T}_{i',j'})} \phi(v)$. Observe that for every $1 \leq i, j \leq \ell$, $G[V_{i,j}]$ is connected. Furthermore for two distinct vertices $x_{i,j}$ and $x_{i',j'}$ we have that $V_{i,j}$ and $V_{i',j'}$ are pairwise disjoint. Now for every edge $e \in \boxplus_\ell$ we will associate a vertex subset. Let $e = x_{i,j}x_{a,b}$. Take the canonical straight line path from $w_{i',j'}$ and $w_{a',b'}$ in Δ_k and let $P_{i',j,a',b'}$ denote the segment of this path that starts after the last vertex from $\mathbb{T}_{i',j'}$ and ends before the first vertex from $\mathbb{T}_{a',b'}$. By our choice of β we have that $P_{i',j,a',b'}$ contains at least 5 vertices. Now with the edge e we associate the vertex subset $V(e) = \cup_{v \in P_{i',j,a',b'}} \phi(v)$. Again observe that for every e we have that $G[V(e)]$ is connected and for any two distinct edges e, e' we have that $V(e) \cap V(e') = \emptyset$. Similarly, for every vertex and edge the corresponding vertex subsets are disjoint.

Let $G^\star$ be the graph obtained by contracting the edges of $S_1, \ldots, S_\ell$. Now using the earlier association we will associate a vertex subsets of $G^\star$ for vertices and edges in $\boxplus_\ell$, which will imply a grid minor of size k in $G^\star$ and hence will imply that the treewidth of $G^\star$ is strictly more than k a contradiction. For a vertex $v \in V(G)$ let $\mathsf{star}(v)$ denote the set of vertices that have been contracted together in $G^\star$. We can similarly define the notion of star of a vertex subset of $V(G)$. Notice that $G^\star[\mathsf{star}(V_{i,j})]$ and $G^\star[\mathsf{star}(V(e))]$ are connected. Since the vertices associated to $x_{i,j}$, where $1 \leq i, j \leq \ell$ are *far apart* we have that $\mathsf{star}(V_{i,j})$ are also pairwise disjoint. For the same reason we can show that for every e_1, e_2, $\mathsf{star}(V(e_1) \cap \mathsf{star}V(e_2) = \emptyset$. However, for an edge

$e = x_{i,j}x_{a,b}$, $\mathsf{star}(V(e)$ could intersect with $\mathsf{star}V_{i,j}$ or $\mathsf{star}V_{a,b}$. Let w_{first} and w_{last} denote the first and the last vertices of $P_{i',j,a',b'}$, respectively. Observe that because of our choice of β, the star of sets associated by ϕ for w_{first} and w_{last} can only intersect with $\mathsf{star}V_{i,j}$ and $\mathsf{star}V_{a,b}$, respectively. This implies that after we remove the vertices of $\mathsf{star}(V_{i,j})$ and $\mathsf{star}(V_{a,b})$ from $\mathsf{star}(V(e))$ we will get a connected component of that has neighbor to both $\mathsf{star}(V_{i,j})$ and $\mathsf{star}(V_{a,b})$. Thus, for an edge e associate the connected component of $\mathsf{star}(V(e)) \setminus (\mathsf{star}(V_{i,j}) \cup \mathsf{star}(V_{a,b}))$ that has neighbor in both $\mathsf{star}(V_{i,j})$ and $\mathsf{star}(V_{a,b})$. This concludes the sketch of the proof. □

In what follows we show how to use Lemma 8.1 for our purposes. We will need the following transformation.

> **Transformation A:** For the transformation we are given a graph G, a weight function $w : V(G) \to \mathbb{Q}^+$ and a solution Z. That is, $Z \subseteq V(G)$ such that $\mathtt{tw}(G/S) \leq \eta$. The graph G remains the same. The weight function w' on $E(G)$ is defined as follows. Fix a spanning forest T of $G[Z]$. Now root each tree of T at some node r. Now orient every edge in T away from the roots. Now for every arc (u, v) in T, we define $w'(uv)$ as $w(v)$. That is, an edge takes the weight of the vertex it points to. For every other edge uv that do not appear in T we set $w'(uv) = \infty$ (that is, some large weight).

Next we describe a simple property of this transformation.

Lemma 8.3. *Let G be a graph excluding an apex graph H as a minor, $w : V(G) \to \mathbb{Q}^+$ be a a weight function, $Z \subseteq V(G)$ be a vertex set. Furthermore, let (G, w') be an instance obtained by applying* **Transformation A** *on G, w and Z. Then, there exists an edge set $S \subseteq E(G)$ such that $\mathtt{tw}(G/S) \leq \mathtt{tw}(G/Z)$ and $w'(S) \leq w(Z)$, and such a set can be obtained from G, Z and w in polyomial time.*

Proof. Let S be the set of all edges e such that $w'(e) \neq \infty$. Then $w'(S) \leq w(Z)$ by construction and $\mathtt{tw}(G/S) = \mathtt{tw}(G/Z)$ because $G/S = G/Z$. □

For us it is important is that a "reverse direction" of Lemma 8.3 also holds when the input graph excludes an apex graph H as a minor, and that this holds for every subset of the edges of finite weight.

Lemma 8.4. *Let G be a graph excluding an apex graph H as a minor, $w : V(G) \to \mathbb{Q}^+$ be a a weight function, and let $Z \subseteq V(G)$ be a vertex set. Furthermore, let (G, w') be the instance obtained by applying* **Transformation A** *on G, w and Z. Then, for every $W \subseteq E(G)$ such that $w'(W)$ is finite there exist a set $C \subseteq V(G)$ such that $w(C) \leq w'(W)$ and $\mathtt{tw}(G/C) \leq c^\star \cdot \mathtt{tw}(G/W)$. Here, $c^\star$ depends on H. Furthermore, there is a polynomial time algorithm to compute C from G,Z,w,w', and W.*

Proof. Let T be the spanning forest defined in **Transformation A**. Since $w'(W)$ is finite we have $W \subseteq E(T)$. We shall abuse notation and refer to W both as an edge set and as a graph. Since $W \subseteq E(T)$ we have that W is a forest. Consider the components $W_1, \ldots, W_\ell$ of W that contain at least one edge. Every tree W_i is a subtree of some rooted tree T_i in the forest T. Let r_i be the vertex in W_i closest to the root of T_i in the tree T_i. Define the vertex set C_i to be $V(W_i) \setminus r_i$, and S_i to be r_i together with the neighbors of r_i in W_i. Let $C = \bigcup_{i=1}^{\ell} C_i$. We claim that C satisfies the statement of the lemma.

First, observe that $w(C) \leq w'(W)$, because for every vertex $c \in C$ the edge from c to the parent of c in T is in W. Second, define $S = \bigcup_{i=1}^{\ell} E(W[S_i])$. We have that $G/W = (G/C)/S$. In particular $S_1, \ldots S_\ell$ are a collection of vertex disjoint stars, and therefore, by Lemma 8.2 we obtain that $\mathtt{tw}(G/C) \leq c_H \cdot (100 \cdot \mathtt{tw}(G/W) + 10)$.

□

Now we are ready to prove Lemma 1.5

Proof of Lemma 1.5. The algorithm takes G, w and U and applies **Transformation A** with $Z = U$. Let w' be the resulting weight function on $E(G)$. By Lemma 8.3 we have that $\mathtt{opt}_{EC}(G, w', t) \le w(U)$. By Lemma 8.3 we obtain an edge set W such that $w'(W) \le d \log n \cdot \mathtt{opt}_{EC}(G, w', t)/s \le d \log n \cdot w(U)/s$ and $\mathtt{tw}(G/W) \le c \cdot st$. Applying Lemma 8.4 on W we obtain a vertex set C such that $w(C) \le d' \cdot w'(W) \le d'' \log n \cdot w(U)/s$ and $\mathtt{tw}(G/C) \le c_H \cdot (100 \cdot \mathtt{tw}(G/W) + 10) \le c_H \cdot (100 \cdot c \cdot st + 10) \le c' \cdot st$. Here choosing $c' = 200 \cdot c_h \cdot c$ suffices. We have proved that C satisfies the statement of the lemma (for the set S), concluding the proof. □

9 Constant factor approximation algorithm for Weighted Treewidth-η Vertex Contraction

In this section we give a constant factor approximation algorithm for the following problem on H-minor free graphs.

Weighted Treewidth-η Vertex Contraction	
Instance:	A graph G, and a weight function $w : V(G) \to \mathbb{Q}^+$.
Objective:	Find a minimum weight set $S \subseteq V(G)$ such that $\mathtt{tw}(G/S) \le \eta$.

Theorem 1.4. *For every fixed constant $\eta \in \mathbb{N}$ and graph H there exists a constant factor approximation algorithm for* Weighted Treewidth-η Vertex Contraction *on H-minor-free graphs. The approximation ratio and the multiplicative constant of the running time of the algorithm depend on H and η, while the degree of the running time does not*[11].

We define $\mathtt{opt}(G, w, \eta)$ to be the (unknown) minimum weight of a subset $U \subseteq V(G)$ such that $\mathtt{tw}(G/U) \le \eta$. Note that $\mathtt{opt}$ is well defined in Theorem 1.4 because there exists a subset $U \subseteq V(G)$ such that $\mathtt{tw}(G/U) \le \eta$ (e.g., select $U = V(G)$). To prove this theorem, we would like to deal with the problem of eliminating large grids as minors rather than a direct computation of treewidth. More precisely, we will focus on the proof of the following lemma.

Lemma 9.1. *Let $\gamma \in \mathbb{N} \setminus \{1\}$ be a fixed constant. There exists a polynomial-time algorithm that given an H-minor free graph G, and a weight function $w : V(G) \to \mathbb{Q}_0^+$, outputs a subset $S \subseteq V(G)$ of weight at most $d \cdot \mathtt{opt}(G, w, \gamma)$ such that G/S has no $\gamma \times \gamma$-grid minor. Here, d is a fixed constant that depends only on H and γ, and $\mathtt{opt}(G, w, \gamma)$ is the (unknown) minimum weight of a subset $U \subseteq V(G)$ such that G/U has no $\gamma \times \gamma$-grid minor. The degree of the polynomial in the running time of the algorithm is independent of H and γ.*

Note that $\mathtt{opt}$ is well defined in Lemma 9.1 because there exists a subset $U \subseteq V(G)$ such that G/U has no $\gamma \times \gamma$-grid minor (e.g., choose $U = V(G)$). Before we turn to prove Lemma 9.1, let us show why it implies Theorem 1.4. Towards this proof we will need the following variant of Courcelle's theorem by Arnborg et al. [3] (see also [15]).

Proposition 9.1 ([3])**.** *Let φ be a formula in monadic second order logic (**MSO**$_2$) with a free variable S. Suppose that we are give a vertex-weighted (possibly boundaried)*[12] *graph G on n vertices, together with a tree decomposition of width t. Moreover, suppose that we are equipped with the evaluation of all the free variables of φ except for X. Then, there exists an algorithm that in time $f(||\varphi||, t) \cdot n$, for some computable function f, finds a subset $S \subseteq V(G)$ of minimum weight such that $\varphi(S)$ is true (if one exists).*

[11] That is, the time complexity is of the form $f(H, \eta) \cdot n^s$ where s is independent of H and η.

[12] We remark that the statement of this proposition works for structures more general than graphs that capture boundaried graphs, that is, our phrasing is an implication of the proposition in its full generality that is suitable for our purpose.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. To distinguish $\texttt{opt}$ and d in Lemma 9.1 from $\texttt{opt}$ and d in the theorem, we refer to the former $\texttt{opt}$ and d by $\texttt{opt}'$ and d', respectively. Given an H-minor free graph G, a weight function $w : V(G) \to \mathbb{Q}_0^+$, we call the algorithm in Lemma 9.1 with $\gamma = \eta + 1$ to obtain a subset $S \subseteq V(G)$. Then, the weight of S can be upper bounded as follows.

$$\begin{aligned} w(S) &\leq d' \cdot \texttt{opt}'(G, w, \eta+1) \\ &\leq d' \cdot \texttt{opt}(G, w, \eta) \end{aligned}$$

Here, the first inequality follows from Lemma 9.1 and because $\gamma = \eta + 1$, and the second inequality follows from Proposition 4.4: since a graph of treewidth at most η cannot have an $(\eta+1)\times(\eta+1)$-grid as a minor (because the treewidth of an $(\eta+1)\times(\eta+1)$-grid is $\eta+1$ and the treewidth of a graph is as least as large as the treewidth of any graph that it has a minor), it follows that $\texttt{opt}'(G, w, \eta+1) \leq \texttt{opt}(G, w, \eta)$.

Now, we consider the treewidth of G/S. Because G/S has no $(\eta + 1) \times (\eta + 1)$-grid as a minor, Proposition 4.3 implies that the treewidth of G/S is smaller than $c' \cdot (\eta + 1) \leq 2c' \cdot \eta$ where c' (which depends on H) is the constant in the proposition.

Let S' be the set of vertices in G/S that are the result of contracting one or more vertices of S. Consider now the instance $(G/S, w', \eta)$ of the Weighted Treewidth-η Vertex Contraction problem where $w'(v) = w(v)$ for every vertex that is not in S', and $w'(v) = 0$ for every vertex in S'. For every subset X of $V(G/S) \setminus S'$ we have that $(G/S)/(S' \cup X) = G/(S \cup X)$. Therefore, since $w'(S') = 0$ we have that $\texttt{opt}(G/S, w', \eta) \leq \texttt{opt}(G, w, \eta)$.

For every fixed η the predicate $\texttt{tw}(G/(S' \cup X)) \leq \eta$ can be formulated in monadic second order logic ($\mathbf{MSO}_2$). Thus, by Proposition 9.1 a set X of minimum weight (with respect to the weight function w') such that $\texttt{tw}((G/S)/(S' \cup X)) \leq \eta$ can be found in time $f(\eta, 2c' \cdot \eta)n$ for some function f. The algorithm outputs the set $X \cup S$ as its solution. We have that $\texttt{tw}(G/(S \cup X)) = \texttt{tw}((G/S)/(S' \cup X)) \leq \eta$, and that

$$w(X \cup S) \leq w(S) + w'(X) \leq (d' + 1) \cdot \texttt{opt}(G, w, \eta)$$

Thus, the statement of theorem is satisfied with $d = d' + 1$. □

The proof of Lemma 9.1 consists of three parts. First, we will show that given an H-minor free graph G of treewidth larger than η, we can find (in polynomial time) an induced subgraph $G[R]$ of G that has a $\gamma \times \gamma$-grid minor but whose treewidth is upper bounded by a function of H and γ, and whose neighborhood in G is upper bounded by a function of H and γ as well. Then, in the second part, we will give a simple rule to process the graph, after which we are able to find a set of vertices in G whose size is upper bounded by a function of H and γ, such that the weight of every vertex in this set can be decreased by $\epsilon > 0$, and as a result of this modification $\texttt{opt}(G, w, \gamma)$ decreases proportionally. Having this result at hand, in the third part we will easily prove Lemma 9.1 by repeatedly applying this result.

9.1 Computing an Induced Subgraph of Moderate Treewidth and Small Neighborhood

The computation in this subsection will rely on the following known result.

Proposition 9.2 ([32])**.** *For any $\epsilon < 1$, there is $\alpha = \alpha(\epsilon)$ such that for any H-minor free graph and $X \subseteq V(G)$ with $\texttt{tw}(G - X) \leq \eta$, there is $X' \subseteq X$ satisfying $|X'| \leq \epsilon|X|$ and for every connected component C of $G - X'$, we have $|V(C) \cap X| \leq \alpha$ and $|N_G(V(C))| \leq \alpha$. Moreover,*

X' can be computed from G and X in polynomial time, where the polynomial is independent of ϵ, α and η.[13]

Having this proposition at hand, we turn to present the computation of the induced subgraph mentioned earlier, stated in the following lemma.

Lemma 9.2. *Let $\gamma \in \mathbb{N}$ be a fixed constant. There exists a polynomial-time algorithm that given an H-minor free graph G such that $\mathtt{tw}(G) \geq p$, outputs a subset $R \subseteq V(G)$ such that the three following properties are satisfied.*

1. *$G[R]$ has a $\gamma \times \gamma$ grid as a minor.*
2. *$\mathtt{tw}(G[R]) \leq q$.*
3. *$|N_G(R)| \leq r$.*

Here, p, q and r are fixed constants that depend only on H and γ. The degree of the polynomial in the running time of the algorithm is independent of H and γ.

Proof. Given H-minor free graph G such that $\mathtt{tw}(G) \geq p$, we compute $R \subseteq V(G)$ as follows. First, we select $\epsilon = \frac{1}{2}$ and $\eta = c \cdot \gamma$, where $c = c(H)$ is the constant in Proposition 4.3. Additionally, fix $p = \eta + 1$, $q = \eta + \alpha$ and $r = \alpha$, where $\alpha = \alpha(\frac{1}{2})$ is the fixed constant in Proposition 9.2. Now, we initialize $X = V(G)$. Then, it clearly holds that $\mathtt{tw}(G - X) = 0 \leq \eta$. Now, we proceed as follows.

1. We apply Proposition 9.2 to obtain (in polynomial time) a subset $X' \subseteq X$ satisfying $|X'| \leq \frac{1}{2}|X|$ and for every connected component C of $G - X'$, we have $|V(C) \cap X| \leq \alpha$ and $|N_G(V(C))| \leq \alpha$.
2. We test whether $\mathtt{tw}(G - X') \leq \eta$ in time $2^{\mathcal{O}(\eta^3 \log \eta)} n$ by using the algorithm of Bodlaender [10] to compute the treewidth of a (general) graph.
3. If $\mathtt{tw}(G - X') \leq \eta$, then update X to X' and return to the first step.
4. Else, there exists a connected component $C^\star$ of $G - X'$ such that $\mathtt{tw}(C^\star) > \eta$. We output $R = V(C^\star)$ and terminate.

In each iteration, the size of X decreases by at least 1 because $|X'| \leq \frac{1}{2}|X|$ and $X \neq \emptyset$ (since $X = \emptyset$ implies that $\mathtt{tw}(G) \leq \eta$ although we know that $\mathtt{tw}(G) \geq p > \eta$). Thus, at most $\mathcal{O}(n)$ iterations are executed, and since each one of them can be executed in polynomial time, the total running time of the algorithm is polynomial in the input size (where the degree of the polynomial is independent of H and γ).

Now, we prove that the output $R = V(C^\star)$ has the three properties in the statement of the lemma. The satisfaction of the third property is immediate because for every connected component C of $G - X'$, we have that $|N_G(V(C))| \leq \alpha$. For the second property, note that $\mathtt{tw}(G - X) \leq \eta$ and hence $\mathtt{tw}(C^\star - X) \leq \eta$, and $|V(C^\star) \cap X| \leq \alpha$. Thus, since $\mathtt{tw}(C^\star) \leq \mathtt{tw}(C^\star - X) + |V(C^\star) \cap X| \leq \eta + \alpha = q$, the second property is satisfied. Lastly, for the first property, recall that $\mathtt{tw}(C^\star) > \eta = c \cdot \gamma$. In turn, by Proposition 4.3, this implies that $G[R]$ has a $\gamma \times \gamma$ grid as a minor. Thus, the proof is complete. □

[13] This result was phrased (in [32]) for any hereditary class of graphs of truly sublinear treewidth with parameter $\lambda > 0$. For us, it is only important to note that the class of H-minor free graphs is a hereditary class of graphs of truly sublinear treewidth with parameter $\frac{1}{2}$.

9.2 Reduction of Weights of Vertices

The purpose of this subsection is to find a subset of vertices in G whose weights can be decreased. Formally, we seek a subset with the properties in the following definition.

Definition 9.1. *Let $\gamma \in \mathbb{N}$, G be a graph, and $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. A subset $X \subseteq V(G)$ is s-*reducible *if it satisfies the following properties, where $\epsilon = \min_{v \in X}\{w(v)\}$.*

1. *$\epsilon > 0$.*
2. *$|X| \leq s$.*
3. *Among all subsets $U \subseteq V(G)$ of minimum weight such that G/U has no $\gamma \times \gamma$-grid minor, there exists at least one whose intersection with X is not empty.*

Additionally, we would like to work with graphs where the set of vertices of weight 0 is an independent set. To this end, for a graph G and a weight function $w : V(G) \to \mathbb{Q}_0^+$, we say that (G, w) is *nice* if $\{v \in V(G) : w(v) = 0\}$ is an independent set in G. Later, we will account for this assumption. Specifically, we will prove the following result.

Lemma 9.3. *Let $\gamma \in \mathbb{N}$ be a fixed constant. There exists a polynomial-time algorithm that given an H-minor free graph G such that (G, w) is nice and $\mathtt{tw}(G) \geq p$, and a weight function $w : V(G) \to \mathbb{Q}_0^+$, outputs a subset $X \subseteq V(G)$ that is s-reducible. Here, p and s are fixed constants that depend only on H and γ. The degree of the polynomial in the running time of the algorithm is independent of H and γ.*

In addition to Lemma 9.2, the proof of Lemma 9.3 will rely on several other results. Specifically, the outline of the rest of this subsection is as follows. First, we will show how to easily ensure that the set vertices of weight 0 is an independent set. Secondly, consider the induced subgraph $G[R]$ and its neighborhood (viewed as a boundaried graph with the neighborhood as the boundary) given by Lemma 9.3. Then, for every subset $\mathcal{F}$ of boundaried graphs of "small" size (and restricted labels) that contains the unboundaried $\gamma \times \gamma$-grid, we will compute a minimum-weight subset of vertices of R and its neighborhood whose contraction results in a boundaried graph whose folio excludes all boundaried graphs in $\mathcal{F}$. Thirdly, we will argue that there exists an optimal solution that selects all of the vertices of at least one of the subsets of minimum weight that we have already computed. Furthermore, we will argue that each of these subsets has at least one vertex whose weight is not 0. Lastly, we will pick one vertex of positive weight from each one of these subsets and argue that this results in a set that is s-reducible.

Eliminating Edges Between 0-Weight Vertices. For a graph G, a weight function $w : V(G) \to \mathbb{Q}_0^+$, and a set $Z = \{v \in V(G) : w(v) = 0\}$, we denote by w_Z the weight function $w_Z : V(G/Z) \to \mathbb{Q}_0^+$ obtained by restricting w to $V(G/Z) \cap V(G)$ and assigning 0 to each vertex in $V(G/Z) \setminus V(G)$ (which originated from the contraction of a connected subset of Z). Then, we have the following simple result. Here, the notation $\mathtt{opt}$ refers to the one defined in Lemma 9.3.

Lemma 9.4. *Let $\gamma, d \in \mathbb{N}$, G be a graph, and $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let $Z = \{v \in V(G) : w(v) = 0\}$. Let $S' \subseteq V(G/Z)$ be a set of weight (by w_Z) at most $d \cdot \mathtt{opt}(G/Z, w_Z, \gamma)$ such that $(G/Z)/S'$ has no $\gamma \times \gamma$-grid minor. Then, $S = (S' \cap V(G)) \cup Z$ is a set of weight (by w) at most $d \cdot \mathtt{opt}(G, w, \gamma)$ such that G/S has no $\gamma \times \gamma$-grid minor.*

Proof. Notice that G/S is a minor of $(G/Z)/S'$ (these two graphs are equal if S' contains all vertices that originated from the contraction of Z). Thus, because $(G/Z)/S'$ has no $\gamma \times \gamma$-grid as a minor, so does G/S. Towards the proof that $w(S) \leq d \cdot \mathtt{opt}(G, w, \gamma)$, let $U \subseteq V(G)$ be a subset of weight $\texttt{opt}(G, w, \gamma)$ such that G/U has no $\gamma \times \gamma$-grid as a minor. Now, observe that $U' = (U \cap V(G/Z)) \cup (V(G/Z) \setminus V(G))$ is a subset of $V(G/U)$ of the same weight as U such that $(G/Z)/U'$ has no $\gamma \times \gamma$-grid as a minor. Therefore, $w_Z(S') \leq d \cdot \texttt{opt}(G/Z, w_Z, \gamma) \leq d \cdot w_Z(U') = d \cdot w(U) = d \cdot \texttt{opt}(G, w, \gamma)$. Since the weight of S is equal to the weight of S', this implies that $w(S) \leq d \cdot \texttt{opt}(G, w, \gamma)$. ☐

Eliminating Subsets of Boundaried Graphs as Minors. Consider the following problem, which we call Boundaried Minor Hitting. Given a boundaried graph G, a weight function $w : V(G) \to \mathbb{Q}_0^+$, and a collection $\mathcal{F}$ of boundaried graphs F with $\bigcup \Lambda(F) \subseteq \bigcup \Lambda(G)$, compute a subset $S \subseteq V(G)$ of minimum weight such that G/S excludes all graphs in $\mathcal{F}$ as minors; if no such subset $S \subseteq V(G)$ exists, then the output should be $\texttt{nil}$. To show that Boundaried Minor Hitting is solvable in time $f(t, \mathcal{F})n^{\mathcal{O}(1)}$, for some computable function f, on graphs of treewidth t, we use Proposition 9.1. We remark that one can also directly design an algorithm for Boundaried Minor Hitting based on standard (though somewhat tedious) dynamic programming on tree decompositions. As for our purpose we just want to obtain any algorithm whose running time is of the form $f(t, \mathcal{F})n^{\mathcal{O}(1)}$ without the need to optimize f, we avoid this.

Lemma 9.5. *The* Boundaried Minor Hitting *problem is solvable in time $f(t, \mathcal{F})n^{\mathcal{O}(1)}$, for some computable function f, on graphs of treewidth t. Here, the degree of n does not depend on t and $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a collection of boundaried graphs. In light of Proposition 9.1, to prove that lemma, it suffices to present a formula φ in $\mathbf{MSO}_2$ with a free variable S such that the size of φ can depend on the size of the encoding of $\mathcal{F}$, and for any graph G, $\varphi(S)$ is true if and only if G/S has no graph in $\mathcal{F}$ as a minor. Note that the size of φ cannot depend on G. Given a boundaried graph F, Fomin et al. [31] presented a formula ψ_F whose size depends on $|V(F)|$ (and the labeling of F) that tests whether F is a minor of G for a given graph G. (To be more precise, their formula refers to a slightly more restricted definition of boundaried graphs where the labeling function assigns integers rather than sets of integers, but it is immediate to see that their formula is extendible to our case.) Having this formula at hand, we can write another formula $\widehat{\psi}_F$ that tests whether F is a minor of G/S for a given graph G and a subset $S \subseteq V(G)$. Then, we define the formula φ simply as $\varphi(S) = \wedge_{F \in \mathcal{F}} \widehat{\psi}(S)$. ☐

We proceed to define that the collection of all sets of boundaried graphs which we can be potentially interested in eliminating by contracting minimum-weight sets of vertices.

Definition 9.2. *Let $\zeta, \tau \in \mathbb{N}$. An* (ζ, τ)-elimination set *is a set $\mathcal{F}$ of boundaried graphs F with $|V(F)| \leq \zeta^2 + \tau$ and $\bigcup \Lambda(F) \subseteq \{1, 2, \ldots, \tau\}$, such that the $\zeta \times \zeta$-grid with empty boundary is a member of $\mathcal{F}$. The* (ζ, τ)-elimination family*, denoted by $\mathsf{Elim}_{\zeta,\tau}$, is the family of all (ζ, τ)-elimination sets.*

As stated in the following lemma, the size of this collection is small.

Lemma 9.6. *Let $\zeta, \tau \in \mathbb{N}$. Then, the maximum size of a (ζ, τ)-elimination set is upper bounded by $2^{\mathcal{O}((\zeta^2+\tau)^2)}$, and $|\mathsf{Elim}_{\zeta,\tau}| \leq 2^{2^{\mathcal{O}((\zeta^2+\tau)^2)}}$. Moreover, $\mathsf{Elim}_{\zeta,\tau}$ can be computed in time $2^{2^{\mathcal{O}((\zeta^2+\tau)^2)}}$.*

Proof. The number of (unboundaried) graphs on at most $\zeta^2 + \tau$ vertices is upper bounded by $2^{(\zeta^2+\tau)^2}$. For each such graph, the number of options in which we can label its vertices with labels in $\{1, 2, \ldots, \tau\}$ to attain a boundaried graph is upper bounded by $(\zeta^2 + \tau + 1)^\tau$ (each label has the choice of which vertex to be assigned to or not to be assigned). Thus, the maximum size of a (ζ, τ)-elimination set is upper bounded by $2^{\mathcal{O}((\zeta + ^2\tau)^2)}$. As $\mathsf{Elim}_{\zeta,\tau}$ is a family of subsets of this set of maximum size, the bound $|\mathsf{Elim}_{\zeta,\tau}| \leq 2^{2^{\mathcal{O}((\zeta^2+\tau)^2)}}$ follows. Furthermore, it should be clear that $\mathsf{Elim}_{\zeta,\tau}$ can be computed in time $2^{2^{\mathcal{O}((\zeta^2+\tau)^2)}}$ by simple enumeration. $\square$

Keeping the objective of the definition of the family $\mathsf{Elim}_{\zeta,\tau}$ in mind, we define a mapping between each one of its (ζ,τ)-elimination sets, say $\mathcal{F}$, to a subset of vertices of minimum-weight in $G[N_G[R]]$ that eliminates all occurrences of boundaried graphs in $\mathcal{F}$ as minors of $G[N_G[R]]$. In a sense that will become clearer in Lemma 9.11, this mapping represents the "restrictions" of solutions to $G[N_G[R]]$. Towards the definition of the mapping, we first need to formally define how to treat $G[N_G[R]]$ as a boundaried graph. We remark that later, the order $<$ in this statement will be arbitrarily chosen.

Definition 9.3. *Let $\gamma \in \mathbb{N}$. Let G be an H-minor free graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. The* boundaried graph of (G,R) *is the graph $B = G[N_G[R]]$ with boundary $\delta(B) = N_G(R)$, label set $\Lambda(B) = \{1,2,\ldots,|N_G(R)|\}$ and labeling $\lambda_B : \delta(B) \to 2^{\Lambda(B)}$ such that for each vertex $v \in \delta(B)$, $\lambda_B(v) = \{i\}$ where $i = |\{u \in \delta(B) : u \leq v\}|$.*

Now, we present the mapping.

Definition 9.4. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let B be the boundaried graph of (G,R). Then, an* elimination mapping *of (G,R) is a function $f : \mathrm{Elim}_{\gamma,|N_G(R)|} \to 2^{V(B)} \cup \{\mathtt{nil}\}$ such that, for every $(\gamma,|N_G(R)|)$-elimination set $\mathcal{F} \in \mathrm{Elim}_{\gamma,|N_G(R)|}$, the following condition is satisfied: among all subsets $S \subseteq V(B)$ such that B/S has no graph in $\mathcal{F}$ as a minor, $f(\mathcal{F})$ is one of minimum weight, where if no such set S exists, then $f(\mathcal{F}) = \mathtt{nil}$.*

Based on Lemmas 9.5 and 9.6, we can efficiently compute this mapping as follows.

Lemma 9.7. *Let $\gamma, q, r \in \mathbb{N}$ be fixed constants. There exists a polynomial-time algorithm that, given a graph G with an order $<$ on $V(G)$, a weight function $w : V(G) \to \mathbb{Q}_0^+$, and a subset $R \subseteq V(G)$ such that $\mathtt{tw}(G[R]) \leq q$ and $|N_G(R)| \leq r$, outputs an elimination mapping of (G,R). The degree of the polynomial in the running time of the algorithm is independent of γ, q and r.*

Proof. The algorithm computes an elimination mapping f of (G,R) as follows. Let B be the boundaried graph of (G,R). For every $(\gamma,|N_G(R)|)$-elimination set $\mathcal{F} \in \mathrm{Elim}_{\gamma,|N_G(R)|}$, it calls the algorithm in Lemma 9.5 with $(B,\mathcal{F})$ as input, and defines $f(\mathcal{F})$ as the set that this call returns. Then, by the correctness of the algorithm in Lemma 9.5 , it is clear that the output f is indeed an elimination mapping of (G,R).

For the running time, observe that a single call to the algorithm in Lemma 9.5 with $(B,\mathcal{F})$ as input takes time $g(\mathtt{tw}(B),\mathcal{F})n^{\mathcal{O}(1)}$, for some computable function g. By Lemma 9.6, in any such call $|\mathcal{F}| \leq 2^{\mathcal{O}((\gamma^2+|N_G(R)|)^2)}$, and it is clear that the total number of bits required to encode $\mathcal{F}$ is upper bounded by $2^{\mathcal{O}((\gamma^2+|N_G(R)|)^2)}$ as well. Thus, because $\mathtt{tw}(G[R]) \leq q$ and $|N_G(R)| \leq r$, we derive that a single call takes time $g'(\gamma,q,r)n^{\mathcal{O}(1)}$ for some computable function g'. Moreover, by Lemma 9.6, we make $|\mathsf{Elim}_{\gamma,|N_G(R)|}| \leq 2^{2^{\mathcal{O}((\gamma^2+|N_G(R)|)^2)}} \leq 2^{2^{\mathcal{O}((\gamma^2+r)^2)}}$ calls. Thus, the total running time of our algorithm is polynomial, where the degree of the polynomial in the running time of the algorithm is independent of γ, q and r. $\square$

Properties of the Representative Mapping. We proceed to prove two properties of the representative mapping. First, we show that each set assigned by the elimination mapping has at least one vertex whose weight is positive, under the assumption that (G,R) is nice and $G[R]$ has a $\gamma \times \gamma$-grid minor.

Lemma 9.8. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Suppose that (G, R) is nice and that $G[R]$ has a $\gamma \times \gamma$-grid as a minor. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let B be the boundaried graph of (G, R). Then, any elimination mapping f of (G, R) satisfies the following condition: for every $\mathcal{F} \in \mathrm{Elim}_{\gamma,|N_G(R)|}$, it holds that $f(\mathcal{F})$ is either* `nil` *or it contains at least one vertex whose weight is positive.*

Proof. Let f be some elimination mapping of (G, R), and let $\mathcal{F} \in \mathrm{Elim}_{\gamma,|N_G(R)|}$ be such that $f(\mathcal{F}) \neq$ `nil`. Denote $S = f(\mathcal{F})$. Because the $\gamma \times \gamma$-grid (with an empty boundary) is a member of $\mathcal{F}$ and $G[R]$ has a $\gamma \times \gamma$-grid minor, the set S must contain at least two adjacent vertices that belong to B.[14] Since the set of vertices of weight 0 is an independent set (because (G, w) is nice), this means that S contains at least one vertex whose weight is positive. $\square$

Secondly, we show that there necessarily exists a solution whose restriction in $N_G[R]$ contains at least one of the sets assigned by the mapping. Towards obtaining this result (stated in Corollary 9.1), we need to prove several claims. We begin by considering the relation between minors in G and minors in a graph G/U obtained by the contraction of some subset $U \subseteq V(G)$ in G.

Observation 9.1. *Let G be a (possibly boundaried) graph with $U \subseteq V(G)$. Let L be a (possibly boundaried) graph. Then, the following two statements are equivalent.*

- *L is a minor of G/U.*
- *There exists a minor model φ for L in G such that, for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$.*

Proof. In one direction, given a minor model φ for L in G/U, define $\varphi' : V(L) \to 2^{V(G)}$ as follows. For every vertex $v \in V(L)$, let $\varphi'(v) = \bigcup_{u \in \varphi(v)} \mathsf{Origin}(u)$. Then, φ' is a minor model for L in G such that, for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \varphi'(v)$ or $T \cap \varphi'(v) = \emptyset$. In the other direction, suppose that we are given a minor model φ for L in G such that, for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$. Then, define $\varphi' : V(L) \to 2^{V(G/U)}$ as follows. For every vertex $v \in V(L)$, let $\varphi'(v) = \{u \in V(G/U) :$ there exists a connected subset $T \subset \varphi(v)$ such that $T = \mathsf{Origin}(u)\}$. Then, φ' is a minor model for L in G/U. $\square$

Now, we argue that the set of "small" minors eliminated by any solution when restricted to $N_G[R]$ belongs to $\mathrm{Elim}_{\gamma,|N_G(R)|}$. Towards this, we first define this set of minors.

Definition 9.5. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let $U \subseteq V(G)$. Then, the* minor set hit by U with respect to (G, R)*, denote by $\mathsf{Hit}(G, R, U)$, is the set of all boundaried graphs F with $|V(F)| \leq \gamma^2 + |N_G(R)|$ and $\bigcup \Lambda(F) \subseteq \{1, 2, \ldots, |N_G(R)|\}$ that are* not *minors of $B/(U \cap V(B))$, where B is the boundaried graph of (G, R).*

We now prove that if G/U has no $\gamma \times \gamma$-grid minor, then the set $\mathsf{Hit}(G, R, U)$ lies in $\mathrm{Elim}_{\gamma,|N_G(R)|}$.

Lemma 9.9. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let $U \subseteq V(G)$ be such that G/U has no $\gamma \times \gamma$-grid minor. Then,* $\mathsf{Hit}(G, R, U) \in \mathrm{Elim}_{\gamma,|N_G(R)|}$.

[14] Note that we cannot claim to have in S two such adjacent vertices that belong to R even though $G[R]$ has a $\gamma \times \gamma$-grid minor.

Proof. Denote $\tau = |N_G(R)|$. Let B be the boundaried graph of (G, R). Denote $\mathcal{F} = \mathsf{Hit}(G, R, U)$, and denote the $\gamma \times \gamma$-grid whose boundary is empty by L. To prove that $\mathcal{F} \in \mathrm{Elim}_{\gamma,|N_G(R)|}$, we need to show that $L \in \mathcal{F}$. In turn, this means that we need to show that $B/(U \cap V(B))$ does not contain L as minor.[15]

Towards the proof, suppose by way of contradiction that $B/(U \cap V(B))$ does contain L as a minor. By Observation 9.1, there exists a minor model φ for L in B such that, for every maximal connected subset T' of $U \cap V(B)$ and for every vertex $v \in V(L)$, either $T' \subseteq \varphi(v)$ or $T' \cap \varphi(v) = \emptyset$. By the definition of a minor model and contraction for boundaried graphs, and because $\delta(L) = \emptyset$, the following statement holds: For every vertex $v \in V(L)$, we have that $\varphi(v) \cap \delta(B) = \emptyset$. Specifically, this means that $\bigcup_{v \in V(L)} \varphi(v) \subseteq R$.

Next, we claim that φ is a minor model for L in G such that, for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$. Since φ is a minor model for L in B and B is a subgraph of G, it is also a minor model for L in G. Now, consider a maximal connected subset T of U. If T is a connected subset of $U \cap V(B)$, then for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$. Now, suppose that T is not a connected subset of $U \cap V(B)$; in particular, $T \setminus V(B) \neq \emptyset$. In this case, we show that $T \cap (\bigcup_{v \in V(L)} \varphi(v)) = \emptyset$. For this purpose, let $T_1, T_2, \ldots, T_r$ be the maximal connected subsets of $T \cap V(B)$ (possibly $r = 0$), and let $T_{r+1} = T \setminus V(B)$. Observe that every subset T_i, $i \in \{1, 2, \ldots, r\}$, must contain at least one vertex from $N_G(R)$ (otherwise T cannot be a connected subset that satisfies $T \setminus V(B) \neq \emptyset$); moreover, for every vertex $v \in V(L)$, either $T_i \subseteq \varphi(v)$ or $T_i \cap \varphi(v) = \emptyset$. Thus, since $(\bigcup_{v \in V(L)} \varphi(v)) \cap N_G(R) = \emptyset$ (because $\bigcup_{v \in V(L)} \varphi(v) \subseteq R$), we derive that for every subset T_i, $i \in \{1, 2, \ldots, r+1\}$, it holds that $T_i \cap (\bigcup_{v \in V(L)} \varphi(v)) = \emptyset$. In turn, this implies that $T \cap (\bigcup_{v \in V(L)} \varphi(v)) = \emptyset$. Thus, our claim regarding φ is correct. However, by Observation 9.1, this implies that G/U contain L as a minor, which contradicts the supposition of the lemma. This completes the proof. $\square$

Towards showing that $U' = (U \setminus N_G[R]) \cup f(\mathsf{Hit}(G, R, U))$ is at least as good as U, we first consider the weight of U'. In light of Lemma 9.9, the implicit assumption in the following lemma concerning the containment of $\mathsf{Hit}(G, R, U)$ in $\mathrm{Elim}_{\gamma,|N_G(R)|}$ (otherwise $f(\mathsf{Hit}(G, R, U))$ is undefined) is valid.

Lemma 9.10. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let f be an elimination mapping of (G, R). Let $U \subseteq V(G)$ be such that G/U has no $\gamma \times \gamma$-grid minor. Then, $f(\mathsf{Hit}(G, R, U)) \neq \texttt{nil}$ and $w(U') \leq w(U)$, where $U' = (U \setminus N_G[R]) \cup f(\mathsf{Hit}(G, R, U))$.*

Proof. Let B be the boundaried graph of (G, R). Notice that, by the definition of $\mathsf{Hit}(G, R, U)$ (Definition 9.5), it holds that $B/(U \cap V(B))$ has no graph in $\mathsf{Hit}(G, R, U)$ as a minor. Thus, it directly follows from the definition of an elimination mapping (Definition 9.4) that $f(\mathsf{Hit}(G, R, U))$ is not `nil`.

Now, observe that $w(U') = w(U \setminus V(B)) + w(f(\mathsf{Hit}(G, R, U)))$ and $w(U) = w(U \setminus V(B)) + w(U \cap V(B))$. Thus, to show that $w(U') \leq w(U)$, it suffices to show that $w(f(\mathsf{Hit}(G, R, U))) \leq w(U \cap V(B))$. For this purpose, recall that $B/(U \cap V(B))$ has no graph in $\mathsf{Hit}(G, R, U)$ as a minor. Since $f(\mathsf{Hit}(G, R, U))$ is a set of minimum weight that has this property (by Definition 9.4), we deduce that $w(f(\mathsf{Hit}(G, R, U))) \leq w(U \cap V(B))$. $\square$

Now, we show that the contraction of U', just like the contraction of U, ensures that we do not have any $\gamma \times \gamma$-grids as minors.

[15]Note that $B/(U \cap V(B))$ might contain some boundaried $\gamma \times \gamma$-grids whose boundary is not empty as minors. We only claim that $B/(U \cap V(B))$ cannot contain L as a minor.

Lemma 9.11. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let f be an elimination mapping of (G, R). Let $U \subseteq V(G)$ be such that G/U has no $\gamma \times \gamma$-grid minor. Then, $f(\mathsf{Hit}(G, R, U)) \neq \mathtt{nil}$ and G/U' has no $\gamma \times \gamma$-grid minor, where $U' = (U \setminus N_G[R]) \cup f(\mathsf{Hit}(G, R, U))$.*

Proof. Denote $\tau = |N_G(R)|$. Let B be the boundaried graph of (G, R). Denote $\mathcal{F} = \mathsf{Hit}(G, R, U)$, and denote the $\gamma \times \gamma$-grid whose boundary is empty by L. As in the proof of Lemma 9.10, $f(\mathcal{F}) \neq \mathtt{nil}$. Now, we proceed to prove that G/U' does not contain L as a minor. For this purpose, suppose by way of contradiction that G/U' contains L as a minor. By Observation 9.1, there exists a minor model φ for L in G such that, for every maximal connected subset T of U' and for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$.

The outline of the rest of the proof is as follows. We will first partition $S \cap V(B)$, for every set S assigned by φ, to the family of every maximal connected subset of $S \cap V(B)$, and argue that this operation results in at most $\gamma^2 + \tau$ sets in total. Then, based on these partitions and the edges that go across them, we will define a boundaried graph $L^\star$ and a minor model $\varphi^\star$ for $L^\star$ in B; in particular the sets assigned by $\varphi^\star$ will be the precisely the collection of sets in our partitions. By applying Observation 9.1, we will deduce that $L^\star$ is a minor of $B/(U' \cap V(B))$. In turn, based on the definition of $f(\mathcal{F})$, we will deduce that $L^\star$ is a minor of $B/(U \cap V(B))$. Applying Observation 9.1 again, we will obtain a minor model $\psi^\star$ for $L^\star$ and B. Lastly, we will "complete" $\psi^\star$ using the vertices in $(\bigcup_{v \in V(L)} \varphi(v)) \setminus V(B)$ to obtain a minor model ψ for L in G. Applying Observation 9.1 one last time, we will derive that G/U contains L as a minor, and thus reach a contradiction.

We begin by defining the partitions. For every vertex $v \in V(L)$, let $\mathrm{cc}_\varphi(v)$ denote the partition of $\varphi(v) \cap V(B)$ into maximal connected sets in B. Now, we upper bound the total number of sets across these partitions.

Claim 9.1. *It holds that $\sum_{v \in V(L)} |\mathrm{cc}_\varphi(v)| \leq \gamma^2 + \tau$.*

Proof. Note that $|V(L)| = \gamma^2$. Thus, to prove that $\sum_{v \in V(L)} |\mathrm{cc}_\varphi(v)| \leq \gamma^2 + \tau$, it suffices to prove that for every vertex $v \in V(L)$, it holds that $|\mathrm{cc}_\varphi(v)| \leq \max\{1, |\varphi(v) \cap N_G(R)|\}$. Indeed, if this claim is true, then

$$\begin{aligned}
\sum_{v \in V(L)} |\mathrm{cc}_\varphi(v)| &= \sum_{\substack{v \in V(L) \\ \text{s.t. } \varphi(v) \cap N_G(R) = \emptyset}} |\mathrm{cc}_\varphi(v)| + \sum_{\substack{v \in V(L) \\ \text{s.t. } \varphi(v) \cap N_G(R) \neq \emptyset}} |\mathrm{cc}_\varphi(v)| \\
&\leq \sum_{\substack{v \in V(L) \\ \text{s.t. } \varphi(v) \cap N_G(R) = \emptyset}} 1 + \sum_{\substack{v \in V(L) \\ \text{s.t. } \varphi(v) \cap N_G(R) \neq \emptyset}} |\varphi(v) \cap N_G(R)| \\
&\leq |V(L)| + |N_G(R)| = \gamma^2 + \tau.
\end{aligned}$$

To prove our claim, consider some vertex $v \in V(L)$. If $\varphi(v) \setminus V(B) = \emptyset$, then $\mathrm{cc}_\varphi(v) = \{\varphi(v)\}$ and hence $|\mathrm{cc}_\varphi(v)| = 1$. Else, every maximal connected subset of $\varphi(v) \cap V(B)$ (possibly no such subset exists) contains at least one vertex in $N_G(R)$, otherwise $G[\varphi(v)]$ cannot be a connected graph; then, $|\mathrm{cc}_\varphi(v)| \leq |\varphi(v) \cap N_G(R)|$. $\diamond$

Next, we define a boundaried graph $L^\star$ as follows:

- $V(L^\star) = \{(v, C) : v \in V(L), C \in \mathrm{cc}_\varphi(v)\}$,
- $E(L^\star) = \{\{(u, C), (v, D)\} : (u, C), (v, D) \in V(L^\star), E_B(C, D) \neq \emptyset\}$,
- $\delta(L^\star) = \{(v, C) \in V(L^\star) : C \cap N_G(R) \neq \emptyset\}$,
- $\Lambda(L^\star) = \{\bigcup\{\lambda_B(u) : u \in C \cap N_G(R)\} : (v, C) \in V(L^\star)\} \setminus \{\emptyset\}$, and

- For every vertex $(v, C) \in \delta(L^\star)$, $\lambda_{L^\star}((v, C)) = \bigcup\{\lambda_B(u) : u \in C \cap N_G(R)\}$.

In particular, Claim 9.1 implies that $L^\star \in \mathrm{Elim}_{\gamma,\tau}$.

Additionally, we define a mapping $\varphi^\star : V(L^\star) \to 2^{V(B)}$ as follows: for every vertex $(v, C) \in V(L^\star)$, define $\varphi^\star((v, C)) = C$. From our definition of $L^\star$, it is immediate that $\varphi^\star$ is a minor model of $L^\star$ in B. Moreover, consider any connected subset T' of $U' \cap V(B)$. Recall that for every maximal connected subset T of U' and for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$. Therefore, for every vertex $v \in V(L)$, either $T' \subseteq C$ for some $C \in \mathrm{cc}_\varphi(v)$ or $T' \cap \varphi(v) = \emptyset$. In particular, this means that for every vertex $(v, C) \in V(L^\star)$, either $T' \subseteq \varphi^\star((v, C))$ or $T' \cap \varphi^\star((v, C)) = \emptyset$. Thus, by Observation 9.1, $L^\star$ is a minor of $B/(U' \cap V(B))$.

As f is an elimination mapping and $U' \cap V(B) = f(\mathcal{F})$, the fact that $L^\star$ is a minor of $B/(U' \cap V(B))$ implies that $L^\star \notin \mathcal{F}$. In turn, because $\mathcal{F} = \mathsf{Hit}(G, R, U)$, we derive that $L^\star$ is a minor of $B/(U \cap V(B))$ as well. By Observation 9.1, there exists a minor model $\psi^\star$ for $L^\star$ in B such that, for every maximal connected subset T of U and for every vertex $(v, C) \in V(L^\star)$, either $T \subseteq \psi^\star((v, C))$ or $T \cap \psi^\star((v, C)) = \emptyset$.

Having $\psi^\star$ at hand, our current goal is to modify $\psi^\star$, by using the vertices in $(\bigcup_{v \in V(L)} \varphi(v)) \setminus V(B)$, to obtain a minor model ψ for L in G such that, for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \psi(v)$ or $T \cap \psi(v) = \emptyset$. To this end, we define a mapping $\psi : V(L) \to 2^{V(G)}$ as follows. For every vertex $v \in V(L)$, define

$$\psi(v) = (\varphi(v) \setminus V(B)) \cup (\bigcup_{C \in \mathrm{cc}_\varphi(v)} \psi^\star((v, C))).$$

We first claim that ψ is a minor model for L in G.[16] To verify this claim, we assert that the three properties of a minor model are satisfied as follows.

1. First, we claim that for all $v \in V(L)$, it holds that $G[\psi(v)]$ is connected. To this end, consider some vertex $v \in V(L)$. Because $\psi^\star$ is a minor mode for $L^\star$ in B and B is a subgraph of G, for all $C \in \mathrm{cc}_\varphi(v)$, we have that $G[\psi^\star((v, C))]$ is connected; furthermore, by the definition of the labeling $\lambda_{L^\star}$, we have that $C \cap N_G(R) = \psi^\star((v, C)) \cap N_G(R)$. If $\psi(v) \setminus V(B) = \emptyset$, then $|\mathrm{cc}_\varphi(v)| = 1$, and we are done. Thus, we next suppose that $\psi(v) \setminus V(B) \neq \emptyset$. Because φ is a minor model for L in G, we have that $G[\varphi(v)]$ is connected. Therefore, as $\varphi(v) \setminus V(B) = \psi(v) \setminus V(B) \neq \emptyset$, for all $C \in \mathrm{cc}_\varphi(v)$, we have that $C \cap N_G(R) = \psi^\star((v, C)) \neq \emptyset$. Therefore, as $\varphi(v) = (\varphi(v) \setminus V(B)) \cup (\bigcup_{C \in \mathrm{cc}_\varphi(v)} C)$, $\bigcup_{C \in \mathrm{cc}_\varphi(v)} \psi^\star((v, C)) \subseteq V(B)$ and $V(B) = N_G[R]$, we conclude that $G[\psi(v)]$ is connected.

2. Second, we claim that for all $u, v \in V(L)$, it holds that $\psi(u) \cap \psi(v) = \emptyset$. To this end, consider some vertices $u, v \in V(L)$. As φ is a minor model, we have that $\varphi(u) \cap \varphi(v) = \emptyset$, which implies that $(\varphi(u) \setminus V(B)) \cap (\varphi(v) \setminus V(B)) = \emptyset$ and $\mathrm{cc}_\varphi(u) \cap \mathrm{cc}_\varphi(v) = \emptyset$. As $\psi^\star$ is a minor model, we have that $\psi^\star((w, C)) \cap \psi^\star((s, D))$ for all $(w, C), (s, D) \in V(L^\star)$. Thus, $(\bigcup_{C \in \mathrm{cc}_\varphi(v)} \psi^\star((u, C))) \cap (\bigcup_{C \in \mathrm{cc}_\varphi(v)} \psi^\star((v, C))) = \emptyset$. Because $\psi^\star((w, C)) \subseteq V(B)$ for all $(w, C) \in V(L^\star)$, we conclude that $((\varphi(u) \setminus V(B)) \cup (\bigcup_{C \in \mathrm{cc}_\varphi(u)} \psi^\star((u, C)))) \cap ((\varphi(v) \setminus V(B)) \cup (\bigcup_{C \in \mathrm{cc}_\varphi(v)} \psi^\star((v, C)))) = \emptyset$, which means that $\varphi(u) \cap \varphi(v) = \emptyset$.

3. Third, we claim that for all $\{u, v\} \in E(L)$, it holds that there exist $u' \in \psi(u)$ and $v' \in \psi(v)$ such that $\{u', v'\} \in E(G)$. To this end, consider some edge $\{u, v\} \in E(L)$. Because φ is a minor model for L in G, we have that there exist $\widehat{u} \in \varphi(u)$ and $\widehat{v} \in \varphi(v)$ such that $\{\widehat{u}, \widehat{v}\} \in E(G)$. When we proved that first property, we have already shown that for all $w \in V(L)$ and $C \in \mathrm{cc}_\varphi(w)$, we have that $C \cap N_G(R) = \psi^\star((w, C)) \cap N_G(R)$.

[16] In this context, we remind that L and G are unboundaried graphs and therefore we deal with the standard definition of a minor model.

Thus, $\varphi(u) \cap N_G(R) = \psi(u) \cap N_G(R)$ and $\varphi(v) \cap N_G(R) = \psi(v) \cap N_G(R)$. Moreover, note that $\varphi(u) \setminus V(B) = \psi(u) \setminus V(B)$. Thus, if $\widehat{u}, \widehat{v} \in (V(G) \setminus R)$, then we are done by choosing $u' = \widehat{u}$ and $v' = \widehat{v}$. Therefore, we next suppose that this is note the case. Then, without loss of generality, suppose that $\widehat{u} \in R$. Because $\{\widehat{u}, \widehat{v}\} \in E(G)$, this implies that $\widehat{v} \in V(B)$. Let C and D be the maximal connected subsets in $\mathrm{cc}_\varphi(u)$ and $\mathrm{cc}_\varphi(v)$ such that $\widehat{u} \in C$ and $\widehat{v} \in D$. As $\{\widehat{u}, \widehat{v}\} \in E(G)$, we have that $\{(u, C), (v, D)\} \in E(L^\star)$ (by our definition of $L^\star$). Because $\psi^\star$ is a minor model for $L^\star$ in B which is a subgraph of G, there exist $u^\star \in \psi^\star((u, C))$ and $u^\star \in \psi^\star((u, C))$ such that $\{u^\star, v^\star\} \in E(G)$. Thus, because $\psi^\star((u, C)) \subseteq \psi(u)$ and $\psi^\star((u, C)) \subseteq \psi(u)$, the proof of claim is complete (select $u' = u^\star$ and $v' = v^\star$).

Now, we claim that for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \psi(v)$ or $T \cap \psi(v) = \emptyset$. For this purpose, consider some maximal connected subset T of U. Then, for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$. Moreover, for every maximal connected subset A of $T \cap V(B)$ and for every vertex $(v, C) \in V(L^\star)$, either $A \subseteq \psi^\star((v, C))$ or $A \cap \psi^\star((v, C)) = \emptyset$. Thus, if $T \subseteq V(G) \setminus V(B)$ or $T \subseteq V(B)$, the proof of our claim is complete (by the definition of $\psi(v)$). Therefore, we next suppose that neither $T \subseteq V(G) \setminus V(B)$ nor $T \subseteq V(B)$. Then, it must hold that $T \cap N_G(R) \neq \emptyset$. Recall that we have already shown, in this case, that every maximal connected subset A of $T \cap V(B)$ contains at least one vertex from $N_G(R)$, and that for all $w \in V(L)$ and $C \in \mathrm{cc}_\varphi(w)$, we have that $C \cap N_G(R) = \psi^\star((w, C)) \cap N_G(R)$. Now, as for every vertex $v \in V(L)$, either $T \subseteq \varphi(v)$ or $T \cap \varphi(v) = \emptyset$, we in particular derive that there exists at most one vertex in $V(L)$, say $v^\star$, such that $T \subseteq \varphi(v)$. For the rest of the proof, we can suppose that $v^\star$ exists, since otherwise $(\bigcup_{v \in V(L)} \varphi(v)) \cap T = \emptyset$ and hence $(\bigcup_{v \in V(L)} \psi(v)) \cap T = \emptyset$ (because $(\bigcup_{v \in V(L)} \varphi(v)) \cap N_G(R) = (\bigcup_{v \in V(L)} \psi(v)) \cap N_G(R)$, and T cannot have non-empty intersection with $\bigcup_{v \in V(L)} \psi(v)$ if it has an empty intersection with $(\bigcup_{v \in V(L)} \psi(v)) \cap N_G(R)$), which completes the proof of the claim. Only the vertex $v^\star$ can have subsets $C \in \mathrm{cc}_\varphi(v^\star)$ such that $T \cap C \neq \emptyset$. Therefore, for every maximal connected subset A of $T \cap V(B)$ and for every vertex $(v, C) \in V(L^\star)$, if $v \neq v^\star$ then $A \cap \psi^\star((v, C)) = \emptyset$, and otherwise either $A \subseteq \psi^\star((v, C))$ or $A \cap \psi^\star((v, C)) = \emptyset$; moreover, for the maximal connected subset $C \in \mathrm{cc}_\varphi(v^\star)$ such that $A \subseteq C$, it holds that $A \subseteq \psi^\star((v^\star, C))$ (because $A \cap \psi^\star((v^\star, C))$ cannot be satisfied as $A \cap N_G(R) = \psi^\star((v^\star, C)) \cap N_G(R) \neq \emptyset$). From this, we conclude that either $T \subseteq \psi(v^\star)$.

So far, we have proved that ψ is a minor model for L in G such that, for every maximal connected subset T of U and for every vertex $v \in V(L)$, either $T \subseteq \psi(v)$ or $T \cap \psi(v) = \emptyset$. Finally, due to this proof and by Observation 9.1, we derive that G/U contains L as a minor. However, recalling that L is the (unboundaried) $\gamma \times \gamma$-grid, this is a contradiction, and therefore the proof is complete. $\square$

From Lemmas 9.10 and 9.11, we directly obtain the following corollary.

Corollary 9.1. *Let $\gamma \in \mathbb{N}$. Let G be a graph with $R \subseteq V(G)$ and an order $<$ on $V(G)$. Let $w : V(G) \to \mathbb{Q}_0^+$ be a weight function. Let f be an elimination mapping of (G, R). Among all subsets $U \subseteq V(G)$ of minimum weight such that G/U has no $\gamma \times \gamma$-grid minor, there exists at least one subset U that satisfies the following condition: There exists a set $\mathcal{F} \in \mathrm{Elim}_{\gamma, |N_G(R)|}$ such that $f(\mathcal{F}) \neq$ `nil` and $f(\mathcal{F}) \subseteq U$.*

Proof of Lemma 9.3. Let p and r be the fixed constants in Lemma 9.2, and $s = |\mathsf{Elim}_{\gamma,r}|$. Given an H-minor free graph G such that (G, w) is nice and $\mathtt{tw}(G) \geq p$, and a weight function $w : V(G) \to \mathbb{Q}_0^+$, the algorithm begins by calling the algorithm in Lemma 9.2 with G as input to compute a subset $R \subseteq V(G)$ such that $G[R]$ has a $\gamma \times \gamma$ grid as a minor, $\mathtt{tw}(G[R]) \leq q$ and $|N_G(R)| \leq r$ for some fixed constant q that depends only on H and γ. Then, the algorithm selects (arbitrarily) some order $<$ on $V(G)$. Next, it calls the algorithm in Lemma 9.7 with (G, w, R) as input (note that all of the conditions required to make this call are satisfied), and thus computes an elimination mapping f of (G, R). For every (γ, r)-elimination set $\mathcal{F} \in \mathsf{Elim}_{\gamma,r}$ such that $f(\mathcal{F}) \neq \mathtt{nil}$, let $x_{\mathcal{F}}$ be some vertex in $f(\mathcal{F})$ of positive weight (whose existence is guaranteed by Lemma 9.8 because (G, w) is nice). Finally, return the set X defined as follows.

$$X = \{x_{\mathcal{F}} : \mathcal{F} \in \mathsf{Elim}_{\gamma,r}, f(\mathcal{F}) \neq \mathtt{nil}\}.$$

The fact that the algorithm runs in polynomial time, where the degree of the polynomial is independent of H and γ, follows directly from the bounds on the running times of the algorithms in Lemmas 9.2 and 9.7.

Now, we turn to prove that X is s-reducible. Let $\epsilon = \min_{v \in X}\{w(v)\}$. The construction of X immediately implies that $|X| \leq s$, and that if X is not empty, then $\epsilon > 0$. By Corollary 9.1, among all subsets $U \subseteq V(G)$ of minimum weight such that G/U has no $\gamma \times \gamma$-grid minor, there exists at least one subset U that satisfies the following condition: There exists a set $\mathcal{F} \in \mathrm{Elim}_{\gamma,|N_G(R)|}$ such that $f(\mathcal{F}) \neq \mathtt{nil}$ and $f(\mathcal{F}) \subseteq U$. For this subset U, it holds that $x_{\mathcal{F}} \in U \cap X$. This inclusion also implies that X is not empty. Thus, the proof is complete. $\square$

9.3 Proof of Lemma 9.1

For the sake of simplicity, we design the algorithm as a recursive algorithm. Let p and s be the fixed constants in Lemma 9.3. Additionally, fix $d = s$. Given an H-minor free graph G, and a weight function $w : V(G) \to \mathbb{Q}_0^+$, our algorithm $\mathsf{ALG}(G, w)$ computes $S \subseteq V(G)$ as follows.

1. **If $\mathtt{tw}(G) \leq p$:** Solve the problem optimally in time $f(p, \gamma)n^{\mathcal{O}(1)}$ by calling the algorithm in Lemma 9.5 on $(G, w, \mathcal{F})$ where G is considered as a boundaried graph with an empty boundary, and the only graph in $\mathcal{F}$ is the $\gamma \times \gamma$-grid (with an empty boundary).

2. **Else if (G, w) is not nice:** Let $Z = \{v \in V(G) : w(v) = 0\}$. Execute a recursive call $\mathsf{ALG}(G/Z, w_Z)$, and let $S' \subseteq V(G/Z)$ be its output. Return $S = (S' \cap V(G)) \cup Z$.

3. **Otherwise:** Call the algorithm in Lemma 9.3 to compute an s-reducible subset $X \subseteq V(G)$. Let $\epsilon = \min_{v \in X} w(v)$, and define $w' : V(G) \to \mathbb{Q}_0^+$ as follows.

$$w'(v) = \begin{cases} w(v) & \text{if } v \notin X \\ w(v) - \epsilon & \text{otherwise} \end{cases}$$

   Execute a recursive call $\mathsf{ALG}(G, w')$, and let $S \subseteq V(G)$ be its output. Then, return S.

For the analysis of the running time, observe that each recursive call decreases the number of vertices that have positive weight. Specifically, in the second case the total number of vertices decreases, and in the third case $\epsilon > 0$ (because X is s-reducible), and therefore at least one vertex in X changes its weight from being positive to 0. Thus, the depth of the recursion is upper bounded by n. Moreover, by Lemma 9.3, the computation in the third case can be done in polynomial time (where the degree of the polynomial is independent of γ and H). Thus, the running time of our algorithm is polynomial (where the degree of the polynomial is independent of γ and H).

Now, let us consider the correctness of the algorithm. Namely, we need to show that the set S returned by $\mathsf{ALG}(G, w)$ has weight at most $d \cdot \mathtt{opt}(G, w, \gamma)$ and that G/S has no $\gamma \times \gamma$-grid minor. The proof is by induction on the number of vertices of positive weight in G. In the basis, $\mathtt{tw}(G) \leq p$. Then, we solve the problem optimally (our problem is the special case of Boundaried Minor Hitting where $\mathcal{F}$ consists only of the $\gamma \times \gamma$-grid), and therefore the inequality holds.

In the step of the induction, we first consider the case where $\mathtt{tw}(G) > p$ and (G, w) is not nice. By the inductive hypothesis, the weight of S' (by w_Z) is at most $d \cdot \mathtt{opt}(G/Z, w_Z, \gamma)$ and $(G/Z)/S'$ has no $\gamma \times \gamma$-grid minor. Then, from Lemma 9.4, we immediately get that $w(S) \leq d \cdot \mathtt{opt}(G, w, \gamma)$ and that G/S has no $\gamma \times \gamma$-grid minor.

Lastly, consider the case where $\mathtt{tw}(G) > p$ and (G, w) is nice. By the inductive hypothesis, we have that $w'(S) \leq d \cdot \mathtt{opt}(G, w', \gamma)$ and G/S has no $\gamma \times \gamma$-grid minor. Since $|X| \leq s$ (because X is s-reducible) and $d = s$, we have that

$$w(S) = w'(S) + \epsilon \cdot |X \cap S| \leq d \cdot \mathtt{opt}(G, w', \gamma) + \epsilon s = d \cdot (\mathtt{opt}(G, w', \gamma) + \epsilon).$$

Thus, to prove that $w(S) \leq d \cdot \mathtt{opt}(G, w, \gamma)$, it suffices to show that $\mathtt{opt}(G, w', \gamma) + \epsilon \leq \mathtt{opt}(G, w, \gamma)$. Because X is s-reducible, there exists a subset $T \subseteq V(G)$ such that $w(T) = \mathtt{opt}(G, w, \gamma)$, G/T has no $\gamma \times \gamma$-grid minor and $X \cap T \neq \emptyset$. The fact that $X \cap T \neq \emptyset$ implies that

$$w'(T) = w(T) - \epsilon |T \cap X| \leq w(T) - \epsilon = \mathtt{opt}(G, w, \gamma) - \epsilon.$$

Note that, because G/T has no $\gamma \times \gamma$-grid minor, it holds that $\mathtt{opt}(G, w', \gamma) \leq w'(T)$. Thus, we derive that $\mathtt{opt}(G, w', \gamma) + \epsilon \leq \mathtt{opt}(G, w, \gamma)$. This completes the proof. □

References

[1] A. Agrawal, D. Lokshtanov, P. Misra, S. Saurabh, and M. Zehavi, *Polylogarithmic approximation algorithms for weighted-f-deletion problems*, in Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX-/RANDOM 2018, August 20-22, 2018 - Princeton, NJ, USA, vol. 116 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018, pp. 1:1–1:15. 2

[2] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows - theory, algorithms and applications*, Prentice Hall, 1993. 33

[3] S. Arnborg, J. Lagergren, and D. Seese, *Easy problems for tree-decomposable graphs*, J. Algorithms, 12 (1991), pp. 308–340. 78

[4] S. Arora, M. Grigni, D. R. Karger, P. N. Klein, and A. Woloszyn, *A polynomial-time approximation scheme for weighted planar graph TSP*, in Proceedings of the Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, 25-27 January 1998, San Francisco, California, USA., ACM/SIAM, 1998, pp. 33–41. 3

[5] V. Bafna, P. Berman, and T. Fujito, *A 2-approximation algorithm for the undirected feedback vertex set problem*, SIAM Journal on Discrete Mathematics, 12 (1999), pp. 289–297. 1, 2

[6] B. S. Baker, *Approximation algorithms for NP-complete problems on planar graphs*, J. ACM, 41 (1994), pp. 153–180. 3

[7] N. Bansal, D. Reichman, and S. W. Umboh, *LP-based robust algorithms for noisy minor-free and bounded treewidth graphs*, in Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, 2017, pp. 1964–1979. 1, 9, 10, 14, 18, 19, 20, 67, 69

[8] J. Baste, I. Sau, and D. M. Thilikos, *Hitting (topological) minors on bounded treewidth graphs*, CoRR, abs/1704.07284 (2017). 2, 6, 7

[9] ——, *Optimal algorithms for hitting (topological) minors on graphs of bounded treewidth*, in 12th International Symposium on Parameterized and Exact Computation, IPEC 2017, September 6-8, 2017, Vienna, Austria, vol. 89 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017, pp. 4:1–4:12. 1, 2, 6, 7

[10] H. L. Bodlaender, *A linear-time algorithm for finding tree-decompositions of small treewidth*, SIAM J. Computing, 25 (1996), pp. 1305–1317. 80

[11] H. L. Bodlaender, M. Cygan, S. Kratsch, and J. Nederlof, *Deterministic single exponential time algorithms for connectivity problems parameterized by treewidth*, in Proceedings of the 40th International Colloquium of Automata, Languages and Programming (ICALP), vol. 7965 of Lecture Notes in Comput. Sci., Springer, 2013, pp. 196–207. 2

[12] G. Călinescu, H. J. Karloff, and Y. Rabani, *An improved approximation algorithm for MULTIWAY CUT*, J. Comput. Syst. Sci., 60 (2000), pp. 564–574. 68

[13] J. Chuzhoy and Z. Tan, *Towards tight(er) bounds for the excluded grid theorem*, in Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, SIAM, 2019, pp. 1445–1464. 2

[14] V. Cohen-Addad, É. C. de Verdière, P. N. Klein, C. Mathieu, and D. Meierfrankenfeld, *Approximating connectivity domination in weighted bounded-genus graphs*, in Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016, 2016, pp. 584–597. 1, 2, 3, 4, 5, 6

[15] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, *Parameterized Algorithms*, Springer, 2015. 11, 21, 78

[16] M. Cygan, M. Pilipczuk, M. Pilipczuk, and J. O. Wojtaszczyk, *An improved FPT algorithm and a quadratic kernel for pathwidth one vertex deletion*, Algorithmica, 64 (2012), pp. 170–188. 1

[17] A. Dawar, M. Grohe, S. Kreutzer, and N. Schweikardt, *Approximation schemes for first-order definable optimisation problems*, LICS 2006, (2006), pp. 411–420. 1, 2, 3

[18] E. D. Demaine, F. V. Fomin, M. Hajiaghayi, and D. M. Thilikos, *Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs*, J. ACM, 52 (2005), pp. 866–893. 3

[19] E. D. Demaine and M. Hajiaghayi, *Bidimensionality: New connections between FPT algorithms and PTASs*, in Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2005), ACM-SIAM, 2005, pp. 590–601. 1, 3

[20] ——, *Linearity of grid minors in treewidth with applications through bidimensionality*, Combinatorica, 28 (2008), pp. 19–36. 8

[21] E. D. Demaine, M. Hajiaghayi, and K. Kawarabayashi, *Contraction decomposition in h-minor-free graphs and algorithmic applications*, in Proceedings of the 43rd ACM Symposium on Theory of Computing, STOC 2011, San Jose, CA, USA, 6-8 June 2011, 2011, pp. 441–450. 3

[22] E. D. Demaine, M. Hajiaghayi, and B. Mohar, *Approximation algorithms via contraction decomposition*, Combinatorica, 30 (2010), pp. 533–552. 3

[23] E. D. Demaine and M. T. Hajiaghayi, *Graphs excluding a fixed minor have grids as large as treewidth, with combinatorial and algorithmic applications through bidimensionality*, in Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2005, Vancouver, British Columbia, Canada, January 23-25, 2005, 2005, pp. 682–689. 1, 11, 21

[24] R. Diestel, *Graph theory*, vol. 173 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, 3rd ed., 2005. 75

[25] D. Eisenstat, P. Klein, and C. Mathieu, *An efficient polynomial-time approximation scheme for Steiner forest in planar graphs*, in Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms (SODA), SIAM, 2012, pp. 626–638. 3

[26] D. Eppstein, *Diameter and treewidth in minor-closed graph families*, Algorithmica, 27 (2000), pp. 275–291. 3

[27] G. Even, J. Naor, S. Rao, and B. Schieber, *Fast approximate graph partitioning algorithms*, SIAM J. Comput., 28 (1999), pp. 2187–2214. 68

[28] J. Fakcharoenphol and K. Talwar, *An improved decomposition theorem for graphs excluding a fixed minor*, in Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques, 6th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems, APPROX 2003 and 7th International Workshop on Randomization and Approximation Techniques in Computer Science, RANDOM 2003, Princeton, NJ, USA, August 24-26, 2003, Proceedings, 2003, pp. 36–46. 68

[29] S. Fiorini, G. Joret, and U. Pietropaoli, *Hitting diamonds and growing cacti*, in Integer Programming and Combinatorial Optimization, 14th International Conference, IPCO 2010, Lausanne, Switzerland, June 9-11, 2010. Proceedings, 2010, pp. 191–204. 1, 2

[30] F. V. Fomin, P. A. Golovach, and D. M. Thilikos, *Contraction obstructions for treewidth*, J. Combinatorial Theory Ser. B, 101 (2011), pp. 302–314. 75

[31] F. V. Fomin, D. Lokshtanov, N. Misra, and S. Saurabh, *Planar F-deletion: Approximation, kernelization and optimal FPT algorithms*, in Proceedings of the 53rd Annual Symposium on Foundations of Computer Science (FOCS), IEEE, 2012, pp. 470–479. 1, 82

[32] F. V. Fomin, D. Lokshtanov, V. Raman, and S. Saurabh, *Bidimensionality and EPTAS*, in Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), SIAM, 2011, pp. 748–759. 79, 80

[33] F. V. Fomin, D. Lokshtanov, and S. Saurabh, *Excluded grid minors and efficient polynomial-time approximation schemes*, J. ACM, 65 (2018), pp. 10:1–10:44. 1, 2, 3, 8

[34] F. V. Fomin, D. Lokshtanov, S. Saurabh, and D. M. Thilikos, *Bidimensionality and kernels*, in Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), SIAM, 2010, pp. 503–510. 1

[35] R. Gandhi, S. Khuller, and A. Srinivasan, *Approximation algorithms for partial covering problems*, J. Algorithms, 53 (2004), pp. 55 – 84. 3

[36] N. Garg, V. V. Vazirani, and M. Yannakakis, *Approximate max-flow min-(multi)cut theorems and their applications*, SIAM J. Comput., 25 (1996), pp. 235–251. 68

[37] M. Grigni, E. Koutsoupias, and C. H. Papadimitriou, *An approximation scheme for planar graph TSP*, in 36th Annual Symposium on Foundations of Computer Science, Milwaukee, Wisconsin, USA, 23-25 October 1995, IEEE Computer Society, 1995, pp. 640–645. 3

[38] M. Grohe, *Local tree-width, excluded minors, and approximation algorithms*, Combinatorica, 23 (2003), pp. 613–632. 3

[39] A. Gupta, E. Lee, J. Li, P. Manurangsi, and M. Wlodarczyk, *Losing treewidth by separating subsets*, in Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, 2019, pp. 1731–1749. 1

[40] D. S. Hochbaum and W. Maass, *Approximation schemes for covering and packing problems in image processing and VLSI*, J. ACM, 32 (1985), pp. 130–136. 3

[41] D. R. Karger, P. N. Klein, C. Stein, M. Thorup, and N. E. Young, *Rounding algorithms for a geometric embedding of minimum multiway cut*, Math. Oper. Res., 29 (2004), pp. 436–461. 68

[42] S. Khanna and R. Motwani, *Towards a syntactic characterization of ptas*, in STOC 1996, ACM, 1996, pp. 329–337. 3

[43] P. N. Klein, *A linear-time approximation scheme for planar weighted TSP*, in 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), 23-25 October 2005, Pittsburgh, PA, USA, Proceedings, 2005, pp. 647–657. 3

[44] ——, *A linear-time approximation scheme for TSP in undirected planar graphs with edge-weights*, SIAM J. Comput., 37 (2008), pp. 1926–1952. 3

[45] P. N. Klein, S. A. Plotkin, and S. Rao, *Excluded minors, network decomposition, and multicommodity flow*, in Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing, May 16-18, 1993, San Diego, CA, USA, 1993, pp. 682–690. 9, 68

[46] J. M. Kleinberg and A. Kumar, *Wavelength conversion in optical networks*, J. Algorithms, 38 (2001), pp. 25–50. 3

[47] E. Lee, *Partitioning a graph into small pieces with applications to path transversal*, in Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, 2017, pp. 1546–1558. 1

[48] R. J. Lipton and R. E. Tarjan, *A separator theorem for planar graphs*, SIAM J. Appl. Math., 36 (1979), pp. 177–189. 3

[49] D. G. Luenberger and Y. Ye, *Linear and Nonlinear Programming*, Springer Publishing Company, Incorporated, 2015. 17

[50] F. Mazoit, *Tree-width of hypergraphs and surface duality*, J. Combinatorial Theory Ser. B, 102 (2012), pp. 671–687. 7, 75

[51] B. Mohar and C. Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, 2001. 75

[52] B. A. Reed, *Tree Width and Tangles: A New Connectivity Measure and Some Applications*, London Mathematical Society Lecture Note Series, Cambridge University Press, 1997, pp. 87—162. 10, 19

[53] N. Robertson and P. D. Seymour, *Graph minors. XIII. The disjoint paths problem*, J. Combinatorial Theory Ser. B, 63 (1995), pp. 65–110. 69

[54] N. Robertson and P. D. Seymour, *Graph minors. xx. wagner's conjecture*, J. Combinatorial Theory Ser. B, 92 (2004), pp. 325–357. 1

[55] I. Sau and D. M. Thilikos, *On self-duality of branchwidth in graphs of bounded genus*, Discrete Applied Mathematics, 159 (2011), pp. 2184–2186. 17

[56] S. Thomassé, *A $4k^2$ kernel for feedback vertex set*, ACM Trans. Algorithms, 6 (2010), pp. 32:1–32:8. 1