Covering Vectors by Spaces in Perturbed Graphic Matroids and Their Duals

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Abstract

Perturbed graphic matroids are binary matroids that can be obtained from a graphic matroid by adding a noise of small rank. More precisely, an r-rank perturbed graphic matroid M is a binary matroid that can be represented in the form $I + P$, where I is the incidence matrix of some graph and P is a binary matrix of rank at most r. Such matroids naturally appear in a number of theoretical and applied settings. The main motivation behind our work is an attempt to understand which parameterized algorithms for various problems on graphs could be lifted to perturbed graphic matroids.

We study the parameterized complexity of a natural generalization (for matroids) of the following fundamental problems on graphs: Steiner Tree and Multiway Cut. In this generalization, called the Space Cover problem, we are given a binary matroid M with a ground set E, a set of *terminals* $T \subseteq E$, and a non-negative integer k. The task is to decide whether T can be spanned by a subset of $E \setminus T$ of size at most k.

We prove that on graphic matroid perturbations, for every fixed r, Space Cover is fixed-parameter tractable parameterized by k. On the other hand, the problem becomes W[1]-hard when parameterized by $r + k + |T|$ and it is NP-complete for $r \leq 2$ and $|T| \leq 2$.

On cographic matroids, that are the duals of graphic matroids, Space Cover generalizes another fundamental and well-studied problem, namely Multiway Cut. We show that on the duals of perturbed graphic matroids the Space Cover problem is fixed-parameter tractable parameterized by $r + k$.

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1 Introduction

In this paper we develop parameterized algorithms on low-rank perturbations of graphic matroids and their duals. These matroids and their matrices naturally appear in various settings. For example, in the emerging Matroid Minors Project of Geelen, Gerards, and Whittle [15], perturbed matroids play a significant role in the characterization of proper minor-closed classes of binary matroids. More precisely, for each proper minor-closed class $\mathcal{M}$ of binary matroids, there exists a nonnegative integer r such that every sufficiently highly connected matroid $M \in \mathcal{M}$, is either a perturbation of graphic or cographic matroid. In other words, there exist matrices $I, P \in \mathrm{GF}(2)^{\ell \times n}$ such that I is the incidence matrix of a graph, the rank of P is at most r, and either M or its dual M^* is represented by $I + P$. Another example of closely related concept is the robust Principal Component Analysis (PCA), a popular approach to robust subspace learning and tracking by decomposing the data matrix into low-rank and sparse matrices. Here data matrix M is assumed to be a superposition of a low-rank perturbation component P and a sparse component I, that is, $M = I + P$. See Candès et al. [4], Wright et al. [24], and Chandrasekaran et al. [5] for further references on robust PCA. In particular, one of the well-studied, see e.g. [22, 26], of the variants of robust PCA is when the structure of the sparse matrix I is imposed from the structure of some graph. Perturbed matroids also come naturally in the settings when a structural input is corrupted by a noise. In graph algorithms, one of the questions studied in the literature about corrupted inputs is—what happens to special graph classes when they are perturbed adversarially? For example, Magen and Moharrami [19], and Bansal, Reichman, Umboh [2], studied approximation algorithms on noisy minor-free graphs, which are the graphs obtained from minor-free graphs by corrupting a fraction of edges and vertices.

Our results. We work with the following classes of binary matroids. A binary matroid M such that M can be represented in the form $I + P$, where I is the incidence matrix of some graph and P is a binary matrix of rank at most r, is called the *r-rank perturbed graphic matroid*. Similarly, when the dual matroid M^* can be represented as $I + P$ for some incidence matrix I and r-rank matrix P, we refer to M as to an *r-rank perturbed cographic matroid.*

In this paper we study parameterized complexity on binary perturbed matroids of the following generic problem. Let us remind that in a matroid M, a set F spans T, denoted by $T \subseteq \mathrm{span}(F)$, if the sets F and $T \cup F$ are of the same rank.

Space Cover **Input:** A binary matroid M with a ground set E, a set of *terminals* $T \subseteq E$, and a non-negative integer k. **Question:** Is there a set $F \subseteq E \setminus T$ with $\lvert F\rvert \leq k$ such that $T \subseteq \mathrm{span}(F)$?

In other words, Space Cover is the problem of covering a given set of vectors T over GF(2) by a minimum-dimension subspace of the space generated by vectors from $E \setminus T$. Space Cover encompasses various problems arising in different domains, such as coding theory, machine learning, and graph algorithms. For example, Space Cover is a natural generalization of Matroid Girth, the problem of finding a minimum set of dependent elements in a matroid. Matroid Girth can be reduced to Space Cover by computing for each element t of M a minimum set of elements of the remaining part of the matroid that covers $T = \{t\}$.

On graphs (equivalently, special classes of binary matroids, namely graphic and cographic matroids), Space Cover generalizes well-studied optimization problems Steiner Tree and Multiway Cut. Various algorithmic techniques were developed for these problems, see e.g. [7], and it is very interesting to see which of these techniques, if any, can be lifted to matroids.

We obtain the following results about the complexity of Space Cover on r-rank perturbed matroids. (In all these results we assume that representation of r-rank perturbed matroid in the form $I + P$ is given.)

- Our first main algorithmic result (Theorem 1) states the following: On r-rank perturbed graphic matroids, for every fixed r, Space Cover is fixed-parameter tractable (FPT) when parameterized by k.
- We also show that a "weaker" parameterization makes the problem intractable. More precisely, we prove that on r-rank perturbed graphic matroids, Space Cover is W[1]-hard when parameterized by $r + k + |T|$ and the problem is NP-complete for $r \leq 2$ and $|T| \leq 2$ (see Theorems 3 and 4 respectively of [14]).
- Our second main algorithmic result (Theorem 2) concerns r-rank perturbed cographic matroids. This theorem states that Space Cover is FPT on r-rank perturbed cographic matroids when parameterized by $r + k$. We find it a bit surprising that the parameterized complexity of Space Cover is different on r-rank perturbed graphic and cographic matroids.

Previous work. Geelen and Kapadia [17] studied the problem of computing the girth of a binary r-rank perturbed matroid. (The girth of a matroid is the length of its shortest circuit.) Geelen and Kapadia have proved that the girth of an r-rank perturbed matroid is fixed-parameter tractable being parameterized by r. Let us note that while Space Cover generalizes Matroid Girth, our results are incomparable. In our FPT result for r-rank perturbed graphic matroids the parameter is k while the parameter r should be fixed. As our complexity lower bounds show, the requirement that r should be fixed and that k should be the parameter are, most likely, unavoidable. For binary matroids, Matroid Girth has several equivalent formulations. For example, it is equivalent to the Minimum Distance problem from coding theory, which asks for a minimum dependent set of columns in a matrix over GF(2). The complexity of this problem was open until 1997, when Vardy showed it to be NP-complete [23]. On the other hand, Geelen, Gerards and Whittle in [16] conjecture that for any proper minor-closed class $\mathcal{M}$ of binary matroids, there is a polynomial-time algorithm for computing the girth of matroids in $\mathcal{M}$. The parameterized version of the problem, namely Even Set, asks whether there is a dependent set $F \subseteq X$ of size at most k. The parameterized complexity of Even Set was a long-standing open question in the area, see e.g. [10], whose complexity was resolved only recently [3].

Space Cover on graphic and cographic matroids is a generalization of Steiner Tree and Multiway Cut, two very well-studied problems on graphs. By the classical result of Dreyfus and Wagner [12], Steiner Tree is fixed-parameter tractable (FPT) parameterized by the number of terminals T. Similar approach can be used to show that Space Cover is FPT on graphic matroids. On cographic matroids Space Cover is equivalent to the Restricted Edge-Subset Feedback Edge Set introduced by Xiao and Nagamochi [25] who also showed that the problem is FPT parameterized by k. Due to its connection to Multiway Cut, the NP-completeness result of Dahlhaus et al. [8] for Multiway Cut with three terminals implies that Space Cover is NP-hard even if $|T| = 3$ on cographic matroids. Fomin et al. in [13] extended the results for Space Cover on graphic and cographic matroids to a more general class of binary matroids, namely, regular matroids, by providing an algorithm of running time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. While the class of regular matroids is a proper minor-class of binary matroids, this class of matroids is incomparable to the class of perturbed matroids. It is also known that Space Cover is hard on general class of binary matroids: By the result of Downey et al. [11], Space Cover is W[1]-hard on binary matroids when parameterized by k even if restricted to the inputs with one terminal.

Organization of the paper. Due to space constraints, we only sketch the proofs of our main algorithmic results (Theorems 1 and 2) in Section 2. The detailed proofs of these theorems and our algorithmic lower bounds (Theorems 3 and 4) are given in the full version of the paper [14]. We conclude in Section 3 by stating some open problems.

2 Overview of Algorithmic Theorems

In this section, we give short descriptions of both of our algorithmic results. For standard graph and matroid-related terms, we refer to the books by Diestel [9] and Oxley [21]. We also give the formal definitions of graph and matroid-related terms in Section 3 of [14].

2.1 Perturbed Graphic Matroids

In this section, we give an overview of the proof of the first main result of the paper. The detailed proof is given in Section 4 of [14] version of the paper.

In this case, r-rank perturbed matroid M is represented by the perturbed incidence matrix $I(G)$ of a (multi) graph G. Formally we define the following problem.

Space Cover on Perturbed Graphic Matroid (Space Cover on PGM)
Input: A (multi) graph G with n vertices and m edges, an $(n \times m)$-matrix P over $GF(2)$ with $\text{rank}(P) \leq r$, a set of *terminals* $T \subseteq E$ where E is the set of columns of the matrix $A = I(G) + P$, and a non-negative integer k.
Question: Is there a set $F \subseteq E \setminus T$ with $|F| \leq k$ such that $T \subseteq \text{span}(F)$ in the binary matroid M represented by A?

▶ **Theorem 1.** *For any fixed constant r,* Space Cover on PGM *is solvable in time $k^{\mathcal{O}(k)} \cdot (n+m)^{\mathcal{O}(1)}$. In particular,* Space Cover on PGM *is* FPT *when parameterized by k whenever r is a constant.*

We underline that r is a constant here, that is, the constants hidden behind the big-O notation in the running time depend on r.

Before proceeding with the overview, it is useful to discuss how Space Cover on PGM is solvable when $r = 0$, i.e. on graphic matroids and what are the main challenges for solving the problem for $r > 0$. On graphic matroids Space Cover corresponds to the following problem. Given a set of terminal edges $T = \{e_1, e_2, \ldots, e_s\}$, we want to find a set of at most k edges $F \subseteq E \setminus T$ such that for every e_i, graph $G[F \cup e_i]$ has a cycle containing e_i. This can be seen as a variant of the Steiner Tree, and more generally, of the Steiner Forest problem. Here we are given a graph G, a collection of pairs of distinct non-adjacent terminal vertices $\{x_1, y_1\}, \ldots, \{x_s, y_s\}$ of G, and a non-negative integer k. The task is to decide whether there is a set $F \subseteq E(G)$ with $|F| \leq k$ such that for each $i \in \{1, \ldots, s\}$, graph $G[F]$ (which we can be assumed to be a forest) contains an (x_i, y_i)-path. The special case when $x_1 = x_2 = \cdots = x_s$, i.e. when edge set F is a tree spanning all demand vertices, is the Steiner Tree problem. To see that Steiner Forest is a special case of Space Cover, we construct the following graph: For each $i \in \{1, \ldots, s\}$, we add a new edge $x_i y_i$ to G.

Denote by G' the obtained graph and let T be the set of added edges and let $M(G')$ be the graphic matroid associated with G'. Then a set of edges $F \subseteq E(G)$ forms a graph containing all (x_i, y_i)-paths if and only if $T \subseteq \text{span}(F)$ in $M(G')$.

Similar to Steiner Tree, Steiner Forest is fixed-parameter tractable parameterized by the number of terminals. This can be shown by applying a dynamic programming algorithm similar to the classical algorithm of Dreyfus and Wagner [12]. Notice that by [14, Theorem 4]), Space Cover on PGM is NP-complete when restricted to the instances with $r \leq 2$ and $|T| \leq 2$. This shows that for our problem the parameterization just by the number of terminals $|T|$ will not work; it also indicates that for matroids we should try a different approach. To show that Steiner Forest is FPT parameterized by the size k of the forest F, one can use the following idea. Since the size of F is at most k, there are $2^{\mathcal{O}(k)}$ non-isomorphic forests, so we can guess the structure of F. In other words, we can guess a forest H on at most k edges such that the solution F to Steiner Forest is isomorphic to H. Thus for each guess of H, the task is reduced to the following constraint variant of Subgraph Isomorphism: For given graph G and forest H, decide whether G contains a forest isomorphic to H and spanning all terminal vertices of G in the prescribed way. This problem can be solved by combining a color coding technique of Alon, Zwick, and Yuster [1] with dynamic programming.

This is exactly the approach we want to push forward for $r > 0$. However in this case reduction to constraint Subgraph Isomorphism is way more difficult. First, while perturbation matrix P is of bounded rank, adding it to $I(G)$ can change an unbounded number of its elements. On the other hand, since the rank of perturbation matrix P is bounded, we know that matrix P contains only a small number of different columns. Thus while adding P to $I(G)$ changes many elements of $I(G)$, the variety of these changes is bounded. We exploit this in order to guess the structure of a solution. Second, for graphic matroids, the way a forest H should be mapped into G is very clear—for every terminal element t, adding t to the solution should create a cycle containing t. This defines the constraints how the edges of the guessed solution should be connected to terminal edges and allows us to reduce the problem to a constraint variant of Subgraph Isomorphism. For $r > 0$, adding P to $I(G)$ completely destroys this nice property of the solution. Interestingly, the bounded rank of perturbation still allows us to establish the constraints expressed as parities of vertex degrees of a small number of vertices in G, coloring of edges of G, and some additional mappings. As a result, by a sequence of reductions, we succeed in reducing the original problem to a version of constraint Subgraph Isomorphism. Due to the nature of constraints, the solution to this problem also requires new ideas on top of color coding and dynamic programming.

We proceed with an overview of the proof of Theorem 1. The proof consists of two main parts. The first part is an FPT-Turing reduction from Space Cover to the following version of Subgraph Isomorphism, which we call Pattern Cover.

Pattern Cover
Input: A (multi) graph G with n vertices and m edges, a non-negative integer t that is a fixed constant, a function $\ell_G : E(G) \to \{1, 2, \ldots, t\}$, a non-negative integer k, a forest H with k vertices, a function $\ell_H : E(H) \to \{1, 2, \ldots, t\}$, a set $U \subseteq V(H)$ and an injective function $f : U \to V(G)$.
Question: Is there an injective homomorphism $g : V(H) \cup E(H) \to V(G) \cup E(G)$ such that (i) for all $e \in E(H)$, it holds that $\ell_H(e) = \ell_G(g(e))$, and (ii) for all $v \in U$, it holds that $g(v) = f(v)$?

In other words, we give a reduction that for an input (G, P, T, k) of Space Cover on PGM in time $k^{\mathcal{O}(k)} \cdot (n+m)^{\mathcal{O}(1)}$ constructs $k^{\mathcal{O}(k)} \cdot (n+m)^{\mathcal{O}(1)}$ instances of Pattern Cover such that (G, P, T, k) is a yes-instance if and only if at least one of the instances of Pattern Cover is.

The second part of the proof is an algorithm for solving Pattern Cover in time $k^{\mathcal{O}(k)} \cdot (n+m)^{\mathcal{O}(1)}$. The combination of the two parts provides the proof of the theorem.

In what follows, we provide a brief description of the FPT-Turing reduction. The reduction is done by a sequence of steps. For simplicity, here we explain how to construct a reduction in time $2^{\mathcal{O}(k^2)} \cdot (n+m)^{\mathcal{O}(1)}$; in Section 4 of [14] we provide more precise arguments that allow to reduce the running time.

We start by bounding $|T|$ by k. In case the columns in T are not linearly independent, we let T' denote a basis of T, and else we denote $T' = T$. We remove the columns in $T \setminus T'$ from $I(G)$ and P, and let (G', P', T', k) denote the resulting instance. Clearly, (G, P, T, k) is a yes-instance if and only if (G', P', T', k) is a yes-instance. Moreover, given a set X of size t of linearly independent vectors, for some $t \in \mathbb{N}$, there does not exist any set Y of vectors of size smaller than t such that $X \subseteq \mathrm{span}(Y)$. Thus, in case $|T'| > k$, the input instance is a no-instance. Therefore, from now onwards we implicitly assume that $|T| \leq k$. We use the term *solution* to refer to any set $F \subseteq E \setminus T$ with $|F| \leq k$ such that $T \subseteq \mathrm{span}(F)$ in the binary matroid M represented by A.

We define $\mathrm{disc}(P) = \{C^1, \ldots, C^t\}$ to be the *set* of the distinct vectors that correspond to the columns in $\{P^e : e \in E(G)\}$ (we index the columns of A, $I(G)$ and P by the edges of G). Since the rank of P is r, it is easy to see that it has at most 2^r different columns, thus $t \leq 2^r$. We say that an edge $e \in E(G)$ is of *type* i, $1 \leq i \leq t$, if $P^e = C^i$ (as vectors). Given an edge $e \in E(G)$, we let $\mathrm{type}(e)$ denote its type. Given a set of edges $E' \subseteq E(G)$, we denote $\mathrm{type}(E', i) = |\{e \in E' : \mathrm{type}(e) = i\}| \mod 2$. Towards to constructing the reduction to Pattern Cover, we define $\ell_G \colon E(G) \to \{1, \ldots, t\}$ by setting $\ell_G(e) = \mathrm{type}(e)$.

We proceed by identifying a small graph that we can guess, and which will guide us how to find a solution. Let F be an inclusion-wise minimal solution; note that the minimality of F implies that F is an independent set. Consider the graph $H = G[\mathrm{edges}(F)]$. The crucial structural lemma that we use states that H is "almost" a forest. More precisely, we show that H has at most 2^t cycles. To see it, assume that H has at least $2^t + 1$ cycles. There are at most t edge types in H. Hence by the pigeonhole principle, there are distinct sets of edges C_1 and C_2 of H that compose cycles and such that $\mathrm{type}(C_1, i) = \mathrm{type}(C_2, i)$ for all $i \in \{1, \ldots, t\}$. Then for the symmetric difference $C = C_1 \triangle C_2$, we obtain that $\mathrm{type}(C, i) = 0$ for $i \in \{1, \ldots, t\}$. Thus the sum of the columns of P corresponding to edges of C is the zero-vector. Notice that since C is the union of cycles of H, the sum of the columns of matrix $I(G)$ corresponding to its edges is also the zero-vector. Hence, the sum of the corresponding vectors of A is also zero; and thus the corresponding set of columns of A, $\{A^e \mid e \in C\} \subseteq F$ is not independent. But this contradicts the minimality of F.

Let $\mathcal{H}$ denote the set of all non-isomorphic graphs with at most k edges, at most 2^t cycles, and no isolated vertices. Thus (G, P, T, k) is a yes-instance of Space Cover on PGM if and only if (G, P, T, k) has a solution isomorphic to some $H \in \mathcal{H}$. It is possible to show that all non-isomorphic graphs in $\mathcal{H}$ can be enumerated within time $2^{\mathcal{O}(k)}$. Therefore, we may explicitly examine each graph $H \in \mathcal{H}$ and check whether we have a solution F with subgraph of G, $G[\mathrm{edges}(F)]$, isomorphic to H. In other words, we are looking for an injective homomorphism $g : V(H) \cup E(H) \to V(G) \cup E(G)$[1] such that $F = \{A^e \mid e \in g(E(H))\}$ is a solution. This is an FPT-Turing reduction which reduces in time $2^{\mathcal{O}(k)}$ the solution of the original problem to the solution of $2^{\mathcal{O}(k)}$ new problems. We will use a less formal term *guess* to refer to such type of reductions. So we guess graph H.

Next, we observe that we can guess the types of edges of H. Since H has at most k edges, there are at most $t^k = 2^{\mathcal{O}(k)}$ distinct functions $\ell_H \colon E(H) \to \{1, \dots, t\}$. Then for each guess of function ℓ_H, we want to decide whether there is an injective homomorphism g such that $\ell_G(g(e)) = \ell_H(e)$ for every $e \in E(H)$ and such that the set of columns F of A corresponding to the image of g, which is $F = \{A^e \mid e \in g(E(H))\}$, is a solution.

By definition, if $F = \{A^e \mid e \in g(E(H))\}$ is a solution, then for each $W \in T$, there is $F_W \subseteq F$ such that

$$W = \sum_{e \in F_W} A^e. \tag{1}$$

(The summations here are modulo 2.) We denote by $E_W = g^{-1}(\text{edges}(F_W))$ the edge subset of H corresponding to F_W. Then by (1),

$$W = \sum_{e \in g(E_W)} (I^e(G) + P^e) = \sum_{e \in g(E_W)} I^e(G) + \sum_{e \in g(E_W)} P^e.$$

Each column P^e is equal to vector $C^{\ell_H(e)}$ from partition $\text{disc}(P)$. Thus

$$W = \sum_{e \in g(E_W)} I^e(G) + \sum_{e \in E_W} C^{\ell_H(e)}. \tag{2}$$

Let $W' = W + \sum_{e \in E_W} C^{\ell_H(e)}$. The rows of matrix $I(G)$ and thus the elements of W' are indexed by the vertices of G. For $v \in V(G)$, we denote by w_v the element of W' indexed by v. Note that w_v is either 0 or 1. Let $V_W = \{v \in V(G) \mid w_v = 1\}$. Observe that V_W is uniquely defined by the choice of W and E_W. The crucial insight, whose proof is given in [14, Section 4], is that (2) and, therefore, (1) holds if and only if g acts as a bijection between V_W and vertices of $H[E_W]$ of odd degrees. This is the most important part of the reduction; it allows to reduce the algebraic requirement that every terminal vector should be in the span of the solution to constraints in the form of bijections, which can be guessed efficiently.

We exploit this property for the next set of guesses. For each $W \in T$, we guess a set $E_W \subseteq E(H)$ and construct V_W as described above. Since $|T| \leq k$ and $|E(H)| \leq k$, we have at most 2^{k^2} possible choices of the sets E_W. Then we find the set $U_W \subseteq V(H[E_W])$ of vertices that have odd degrees in $H[E_W]$. If $|V_W| \neq |U_W|$, we discard the choice. Otherwise, we set $U = \cup_{W \in T} U_W$. Notice that if our guesses correspond to a (potential) solution F, we have that corresponding injective homomorphism g should map U to $V' = \cup_{W \in T} V_W$ bijectively and, moreover, g should act as bijection between each U_W and V_W. We make all possible guesses of a bijection $f \colon U \to U'$. Since $|U| \leq 2k$, we have at most $(2k)^{2k}$ possible choices. Then for each U and f, we are searching for an injective homomorphism $g : V(H) \cup E(H) \to V(G) \cup E(G)$ such that (i) for all $e \in E(H)$, $\ell_H(e) = \ell_G(g(e))$, and (ii) for each $v \in U$, $g(v) = f(v)$.

Now we are ready for the final step of our reduction. Recall that H in the statement of Pattern Cover is required to be a forest. The graph H that was guessed so far does not have this property, but it is "almost" a forest, that is, it has at most 2^t cycles. To fix it, we

[1] Since we handle *multi* graphs, we define the domain and image of g to include edge-sets.

guess a set of edges $S \subseteq E(H)$ of size at most 2^t such that the graph obtained from H by the deletion of S is a forest and set $H = H - S$. Since $|S| \leq 2^t$, and t is a constant depending on r only, we can make a polynomial number of guesses how solution g could map S to $E(G)$; we have at most $|E(G)|^{2^t} = m^{\mathcal{O}(1)}$ possibilities for such partial mappings. For each guess of mapping $h : S \to V(G)$, we modify U and f respectively. Namely, we set $U = U \cup V(H[S])$ and define $f(v) = h(v)$ for $v \in V(H[S])$ as prescribed by our choice of the mapping h of S.

This concludes the description of the construction of an instance of Pattern Cover. It is possible to show that (G, P, T, k) is a yes-instance of Space Cover on PGM if and only if for at least one of the described guesses of a forest H, functions ℓ_H, ℓ_G, set $U \subseteq V(H)$ and function $f : U \to V(G)$, the instance of Pattern Cover with these parameters is a yes-instance. Since the total number of guesses we make is , $2^{\mathcal{O}(k^2)} \cdot (n+m)^{\mathcal{O}(1)}$, our construction is the required FPT-Turing reduction.

In order to solve Pattern Cover, and to complete the proof of Theorem 1, we still have to solve Pattern Cover. This is done by a non-trivial application of the color coding technique combined with dynamic programming. We give all the details in [14, Section 4].

2.2 Duals of Perturbed Graphic Matroids

In this section, we give an overview of the proof of our second main result. The detailed proof of the theorem is given in [14, Section 5].

Formally, we define the following problem.

Space Cover on Dual of Perturbed Graphic Matroid (Space Cover on Dual-PGM)
Input: A (multi) graph G with n vertices and m edges, an $(n \times m)$-matrix P over $GF(2)$ with $\text{rank}(P) \leq r$, a set of *terminals* $T \subseteq E$ where E is the set of columns of the matrix $A = I(G) + P$, and a non-negative integer k.
Question: Is there a set $F \subseteq E \setminus T$ with $|F| \leq k$ such that $T \subseteq \text{span}(F)$ in the dual M^* of the binary matroid M represented by A?

► **Theorem 2.** Space Cover on Dual-PGM *is solvable in time* $2^{2^{\mathcal{O}((2^r+k^2)k)}} \cdot (n+m)^{\mathcal{O}(1)}$. *In particular,* Space Cover on Dual-PGM *is* FPT *when parameterized by* $r + k$.

As in the case with graphic matroids, it is useful to recall how Space Cover on Dual-PGM is solvable for $r = 0$, i.e. on cographic matroids. In a cographic matroid a circuit corresponds to a cut in the underlying graph G. In this case the solution set F should satisfy the following property: for every terminal element $e \in T$ there is a partition (or a cut) $(X_e, \overline{X}_e)$ of the vertex set of G such that this cut, i.e. the set of edges between X_e and $\overline{X}_e$, is of the form $\{e\} \cup F_e$, where $F_e \subseteq F$. Thus e is the only edge in the cut from T and all other edges are from F.

In graph theory this problem is known under name Edge Subset Feedback Edge Set. Xiao and Nagamochi [25] showed that this problem is FPT parameterized by $k = |F|$. The algorithm for solving Edge Subset Feedback Edge Set, as well as its special case Multiway Cut, uses the technique of Marx based on important separators [20]. The essence of this technique is that all required information about the cuts in a graph can be extracted from a carefully selected set of separators of size at most k. However, we do not see how this approach can be shifted to more general matroids, even when the rank of perturbation matrix is 1. The difficulty in this case is that solution F together with T cannot be represented as the union of the sets of edges of cuts in G anymore, and thus the sizes of important separators in G cannot be bounded by a function of k only. In order to overcome this challenge, we have to apply more powerful method of recursive understanding [6].

On a general level, the structure of the proof of Theorem 2 is similar to the structure of the proof of Theorem 1. It consists of two parts. In the first part we give FPT-Turing reduction to a cut problem on graphs and in the second part we use the method of recursive understanding to solve the problem. But here the similarities end. While on perturbation of graphic matroids SPACE COVER is about subgraph isomorphisms, on perturbation of cographic matroids it is about collections of cuts in graphs. This makes both parts of the proof of Theorem 2 much more challenging than in Theorem 1. In order to introduce the graph-cut problem we reduce to, we need several definitions.

Graph problem. Let G be a graph with n vertices and m edges given together with a set of terminal edges T and a partition of $V(G) = (V_1, V_2, \ldots, V_t)$. In addition, for every $e \in T$ graph G is provided with a function $f_e : E(G) \to \{0, 1\}$ and a binary vector $B^e = (b^e_1, b^e_2, \ldots, b^e_t)$.

For terminal edge $e \in T$ and a partition $(X, \bar{X})$ of $V(G)$, we say that an edge $e' \in E(G)$ *contributes to* $(e, (X, \bar{X}))$ (with respect to f_e) if one of the following conditions holds

1. Both endpoints of e' belong to X and $f_e(e') = 1$.
2. Both endpoints of e' belong to $\bar{X}$ and $f_e(e') = 1$.
3. Exactly one of the endpoints of e' belongs to X and $f_e(e') = 0$.

Accordingly, we define $\text{contribute}(e, X)$ as the set of edges that contribute to $(e, (X, \bar{X}))$.

For partition $(X, \bar{X})$ of $V(G)$, and terminal edge $e \in T$, we say that $(X, \bar{X})$ *almost fits* e (with respect to f_e) if $T \cap \text{contribute}(e, X) = \{e\}$. Moreover, if $(X, \bar{X})$ almost fits e and for all $1 \le i \le t$, it holds that $|X \cap V_i| = b^e_i \mod 2$, then we say that $(X, \bar{X})$ fits e (with respect to f_e and B^e).

We are now ready to define our graph problem.

> EDGE-SET COVER
> **Input:** A (multi) graph G with n vertices and m edges, non-negative integers k and t, a partition $(V_1, V_2, \ldots, V_t)$ of $V(G)$, a set $T \subseteq E(G)$, a binary vector $B^e = (b^e_1, b^e_2, \ldots, b^e_t)$ for $e \in T$, and a function $f_e : E(G) \to \{0, 1\}$ for $e \in T$.
> **Question:** Is there a set $F \subseteq E(G) \setminus T$ with $|F| \le k$ such that for each $e \in T$, there exists a partition $(X_e, \bar{X}_e)$ of $V(G)$ that fits e and such that $\text{contribute}(e, X_e) \setminus \{e\} \subseteq F$?

In other words, we a looking for a set of edges F of size k, such that for every terminal edge e, there is a cut $(X_e, \bar{X}_e)$ such that (i) the parities of the intersections of X_e with sets V_i constitute vector B^e, (ii) e is the only terminal edge contributing to the cut and all other edges contributing to the cut are from F.

In the first part of the proof we give a reduction that for an input (G, P, T, k) of SPACE COVER ON DUAL-PGM in time $2^{\mathcal{O}(k2^r)} \cdot (n+m)^{\mathcal{O}(1)}$ constructs $2^{\mathcal{O}(k2^r)} \cdot (n+m)^{\mathcal{O}(1)}$ instances of EDGE-SET COVER such that (G, P, T, k) is a yes-instance if and only if at least one of the instances of EDGE-SET COVER is.

As in the case of perturbed graphic matroids, we can assume that $|T| \le k$. Let $\text{disr}(P) = \{R_1, \ldots, R_t\}$ be the set of the distinct vectors corresponding the rows of P. Since the rank of P is r, it has at most 2^r different rows, hence $t \le 2^r$. Accordingly, we say that a vertex $v \in V(G)$ is of *type* i, $1 \le i \le t$, if $P_v = R_i$. Given a vertex $v \in V(G)$, we let $\text{type}(v)$ denote its type. For $i \in \{1, \ldots, t\}$, we denote by V_i the set of vertices of type i.

Characterization of solutions. For SPACE COVER ON DUAL-PGM, we use the term *solution* to refer to a set $F \subseteq E \setminus T$ with $|F| \le k$ such that $T \subseteq \text{span}(F)$ in the dual M^* of the binary matroid M represented by A. Let I be a binary vector with m elements. Recall that given $F \subseteq E$, edges(F) denotes the set of all edges $e \in E(G)$ such that $A^e \in F$. Now, given a set $F \subseteq E$, we say that I is the *characteristic vector* of F if the i^{th} entry of I is 1 if and only if F contains the i^{th} column of A. Moreover, a set $F \subseteq E$ is a *cocycle* in M if and only if it is a cycle in M^*. We need the following folklore result (see, e.g., [17]) characterizing cocycles of binary matroids.

▶ **Proposition 3.** *Let M be a binary matroid represented by an $(n \times m)$-matrix A, and let F be a subset of E, where E is the set of columns of A. Then, F is a cocycle in M if and only if the characteristic vector of F belongs to* $\text{span}(V)$*, where V is the set of rows of A.*

Note that a set $F \subseteq E \setminus T$ is a solution if and only if for each terminal $W \in T$, there exists a subset $F_W \subseteq F$ such that $F_W \cup \{W\}$ is a cocycle in M. Thus, in light of Proposition 3, we can think of a solution as follows:

▶ **Observation 4.** *A set $F \subseteq E \setminus T$ is a solution if and only if $|F| \leq k$ and for each terminal $W \in T$, there exists a subset $F_W \subseteq F$ such that the characteristic vector of $F_W \cup \{W\}$ belongs to* $\text{span}(V)$*, where V is the set of rows of A.*

Let F be a solution. For each $W \in T$, denote by $e(W)$ the edge of G corresponding to the terminal W. By Observation 4, for each $W \in T$, there is $F_W \subseteq F$ such that the characteristic vector I_W of $F'_W = F_W \cup \{W\}$ belongs to $\text{span}(V)$. It means that there is a set of vertices $X_{e(W)} \subseteq V(G)$ such that $I_W = \sum_{v \in X_{e(W)}} A_v$. Hence, for each $W \in T$, we have the corresponding partition $(X_{e(W)}, \bar{X}_{e(W)})$ of $V(G)$, and the solution can be represented as a collection of cuts $\{(X_{e(W)}, \bar{X}_{e(W)}) \mid W \in T\}$ of G.

For each $W \in T$ and $i \in \{1, \ldots, t\}$, we guess the parity of $|X_{e(W)} \cap V_i|$ and define the vector $B^{e(W)} = (b_1^{e(W)}, \ldots, b_t^{e(W)})$ respectively by setting $b_i^{e(W)} = |X_{e(W)} \cap V_i| \mod 2$. Notice that we have at most 2^{tk} choices for $B^{e(W)}$, because $|T| \leq k$. For each guess, we are now looking for a solution represented by a collection of cuts $\{(X_{e(W)}, \bar{X}_{e(W)}) \mid W \in T\}$ of G such that $|X_{e(W)} \cap V_i| \mod 2 = b_i^{e(W)}$ for $W \in T$ and $i \in \{1, \ldots, t\}$.

Let $I_W = \sum_{v \in X_{e(W)}} A_v$ and let i_e^W for $e \in E(G)$ denote the elements of I_W. We have that

$$I_W = \sum_{v \in X_{e(W)}} (I_v(G) + P_v) = \sum_{v \in X_{e(W)}} I_v(G) + \sum_{v \in X_{e(W)}} P_v. \tag{3}$$

Let $P_W = \sum_{v \in X_{e(W)}} P_v$. Since $|X_{e(W)} \cap V_i| \mod 2 = b_i^{e(W)}$ for $W \in T$ and $i \in \{1, \ldots, t\}$, we obtain that $P_W = \sum_{i=1}^{t} b_i^{e(W)} R_i$. Notice that vector P_W is uniquely defined by the choice of $B^{e(W)}$. We define $f_{e(W)} \colon E(G) \to \{0, 1\}$, by setting $f_{e(W)}(e)$ to be equal to the element of P_W corresponding to e.

Recall that I_W is the characteristic vector of the cocycle F'_W. It means that $A^e \in F'_W$ if and only if $i_e^W = 1$. By making use of (3), we are able to show that for each edge $e \in E(G)$, $A^e \in F'_W$ if and only if one of the following holds:

- Both endpoints of e belong to $X_{e(W)}$ and $f_{e(W)}(e) = 1$.
- Both endpoints of e belong to $\bar{X}_{e(W)}$ and $f_{e(W)}(e) = 1$.
- Exactly one of the endpoints of e belongs to $X_{e(W)}$ and $f_{e(W)}(e) = 0$.

We have that $W \in F'_W$ and W is the unique element of T in this set. It means that for edge $e(W)$, cut $(X_{e(W)}, \bar{X}_{e(W)})$ almost fits $e(W)$ with respect to $f_{e(W)}$. Since $|X_{e(W)} \cap V_i| \mod 2 = b_i^{e(W)}$ for each $W \in T$ and $i \in \{1, \ldots, t\}$, we have that $(X_{e(W)}, \bar{X}_{e(W)})$ fits $e(W)$ with respect to $f_{e(W)}$ and $B^{e(W)}$. Moreover, we prove that for each $W \in T$ and $i \in \{1, \ldots, t\}$ the following are equivalent

- F'_W is a cocycle of M such that $|X_{e(W)} \cap V_i| \mod 2 = b_i^{e(W)}$ and such that its characteristic vector is expressible as $I_W = \sum_{v \in X_{e(W)}} A_v$;
- Cut $(X_{e(W)}, \bar{X}_{e(W)})$ fits $e(W)$ with respect to $f_{e(W)}$ and $B^{e(W)}$.

We also have that $F'_W = \text{contribute}(e(W), X_{e(W)})$.

Now we can complete the reduction to Edge-Set Cover. We consider the partition $(V_1, \dots, V_t)$ of $V(G)$ and the set of terminal edges $T_G = \{e(W) \mid W \in T\}$. For each $W \in T$, we have a binary vector $B^{e(W)}$ and a function $f_{e(W)}$. Together, all these parameters compose an instance of Edge-Set Cover.

Solving Edge-Set Cover**.** The algorithm for Edge-Set Cover is the most technical part of the paper. Here we briefly highlight the approach. On a high-level, we use the method of recursive understanding [6], in which we incorporate various new, delicate subroutines. Informally, this means that at the basis, we are going to deal with a "highly-connected" or a small graph, and at each step where our graph is not highly-connected, we will break it using a very small number of edges into two graphs that are both neither too small nor too large.

Let G be a connected graph, and let p and q be positive integers. A partition (X, Y) of $V(G)$ is called (q, p)-*good edge separation* if $|X|, |Y| > q$, $|E(X, Y)| \leq p$, and $G[X]$ and $G[Y]$ are connected graphs.

Roughly speaking, a graph G is unbreakable if every partition of $V(G)$ with few edges going across must contain a large chunk of $V(G)$ in one of its two sets. Intuitively, this means that G is "highly-connected": any attempt to "break" it severely by using only few edges is futile. Formally, a graph G is (q, p)-*unbreakable* if it does not have a (q, p)-*good edge separation.*

If a graph G is not (q, p)-unbreakable, we say that it is (q, p)-breakable. Chitnis et al. [6] proved the following result.

▶ **Proposition 5** ([6]). *There exists a deterministic algorithm that given a connected graph G along with integers q and p, in time $\mathcal{O}(2^{\min\{q,p\} \cdot \log(q+p)} \cdot (n+m)^3 \log(n+m))$ either finds a (q, p)-good edge separation, or correctly concludes that G is (q, p)-unbreakable.*

In our case, we set $p = 2(k+1)$ and $q = 2^{2^{\lambda(t+k^2)|T|}}$ for some appropriate constant λ. To apply the method of recursive understanding, we introduce a special variant of Edge-Set Cover called Annotated Edge-Set Cover (see [14, Section 5] for the formal definition) that is tailored to apply recursion. We show that we can assume that the input graph G is connected. If G has bounded (by some function of r and k) size, we solve Annotated Edge-Set Cover directly. Otherwise, we use Proposition 5 to check whether G is (q, p)-unbreakable.

If G is not (q, p)-unbreakable, we find a (q, p)-good separation (X, Y) of G. Then we solve a special instance of Annotated Edge-Set Cover for one of the graphs $G[X]$ and $G[Y]$ recursively. We use the obtained solution to construct a new instance of the problem for a graph G' that has less vertices than G. Then we call our algorithm for this smaller instance.

If G is (q, p)-unbreakable, we obtain the crucial basic case that we briefly discuss here. For simplicity, we consider this case for Edge-Set Cover.

Recall that in the definition of Edge-Set Cover, we ask about a set $F \subseteq E(G) \setminus T$ with $|F| \leq k$ such that for each $e \in T$, there exists a partition $(X_e, \bar{X}_e)$ of $V(G)$ that fits e and such that $\text{contribute}(e, X_e) \setminus \{e\} \subseteq F$. We relax these conditions and look for a collection of partitions $\{(Y_e, \bar{Y}_e) \mid e \in T\}$ such that $(Y_e, \bar{Y}_e)$ almost fits e and $|\text{contribute}(e, Y_e) \setminus \{e\}| \leq k$ for $e \in T$. Then we can find such an auxiliary collection of partitions $\{(Y_e, \bar{Y}_e) \mid e \in T\}$ by reducing the relaxed problem to at most k instances of the Edge Odd Cycle Transversal problem (also known as Edge Bipartization). The latter problem could be solved by the results of Guo et al. [18]. Finally we use auxiliary partitions $\{(Y_e, \bar{Y}_e) \mid e \in T\}$ to construct the required collection of partitions $\{(X_e, \bar{X}_e) \mid e \in T\}$ and a set F of size at most k. The final construction heavily exploit the high connectivity of G which allows to search only a "small neighborhood" of $(Y_e, \bar{Y}_e)$.

3 Conclusion

In this paper we established the fixed-parameter tractability of Space Cover on PGM and Space Cover on Dual-PGM. We also know that on the class of binary matroids Space Cover is not tractable. So where lies the tractability border for Space Cover? Our positive results on perturbed matroids, combined with the structure theorem of Geelen, Gerards, and Whittle [16], rise a natural question: could the tractability of Space Cover be extended to any proper minor-closed class $\mathcal{M}$ of binary matroids? Let us note that while we formulate Space Cover only on binary matroids, it can be naturally defined on any class of matroids. In particular, the parameterized complexity of Space Cover on proper minor-closed classes of matroids representable over a finite field is open.

Finally, two concrete open questions. First, what is the parameterized complexity of Space Cover on PGM when $|T|$ is a constant and the parameter is $r + k$? Second, we know that Space Cover on PGM is NP-complete even when $|T| = 2$ and $r \leq 2$ (see [14, Theorem 4]). On the other hand, for $r = 0$ the problem is in P for any fixed number of terminals (it is actually FPT parameterized by $|T|$). What about the case $|T| = 2$ and $r = 1$?

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