Gehrlein Stable Committee with Multi-Modal Preferences

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Abstract. Inspired by Gehrlein stability in multiwinner election, in this paper, we define several notions of stability that are applicable in multiwinner elections with multimodal preferences, a model recently proposed by Jain and Talmon [ECAI, 2020]. In this paper we take a two-pronged approach to this study: we introduce several natural notions of stability that are applicable to multiwinner multimodal elections (MME) and show an array of hardness and algorithmic results.
In a multimodal election, we have a set of candidates, $\mathcal{C}$, and a multi-set of ℓ different preference profiles, where each profile contains a multi-set of strictly ordered lists over $\mathcal{C}$. The goal is to find a committee of a given size, say k, that satisfies certain notions of stability. In this context, we define the following notions of stability: global-strongly (weakly) stable, individual-strongly (weakly) stable, and pairwise-strongly (weakly) stable. In general, finding any of these committees is an intractable problem, and hence motivates us to study them for restricted domains, namely single-peaked and single-crossing, and when the number of voters is odd. Besides showing that several of these variants remain computationally intractable, we present several efficient algorithms for certain parameters and restricted domains.

Keywords: Multiwinner Election · Multi-modal· Stability · Parameterized Complexity.

1 Introduction

In social choice theory, *multiwinner election* is an important problem as many real-life problems such as the selection of the members of a Parliament, research papers for a conference, restaurant menu, a team of players for a team sports competition, a catalogue of movies for an airline, locations for police or fire stations in a city, etc., can be viewed as a "multiwinner election" problem. Mathematically modelled as a problem where the input consists of a set of alternatives (called candidates), a set of voters such that every voter submits a ranking (a total order) over the candidates, called the preference list of the voter[6], and a positive integer k. The goal is to choose a k-sized subset of candidates (called a *committee*) that satisfy certain acceptability conditions.

This model has an obvious limitation in that in real-life scenarios, rarely does one factor decide the desirability of a subset of candidates. In fact, in complex decision making scenarios, e.g., selecting research papers for a conference, a team of astronauts for a space mission, hors-d'oeuvres for a banquet, a team of players for a basketball competition, a catalogue of movies for an airline, etc., multiple competing factors (call them attributes) come into play. Certain candidates may rank highly with respect to some attributes and lowly with respect to others. In choosing a solution, the goal is to balance all these factors and choose a committee that scores well on as many factors as possible. In our modelling of the committee selection problem, the multiple attributes under consideration can be modelled by submitting ℓ different preference profiles, where each profile is a set of strict rankings of the candidates based on a specific attribute. Such a model has been studied in [9,26,34] and the importance of such a model is also highlighted in [4]. How we aggregate all these information to produce a high quality solution with desirable properties is the context of this work. We use the term Multimodal Committee Selection (as opposed to unimodal setting where $\ell = 1$), introduced by Jain and Talmon [26], to refer to the problem under consideration.

Of the many notions of a good solution, the one that comes readily to mind is the one closely associated to "popularity", i.e., a solution that is preferred by at least half of the voters, known as the Condorcet winner. Fishburn [21] generalized Condorcet's idea for a single winner election (when $k = 1$) to a multiwinner election (when $k > 1$). Darmann [12] defined two notions of a Condorcet committee: *weak* and *strong*, where the ranking over the committees is based on some *scoring rules*. Gehrlein [24] proposed a new notion of a Condorcet committee that compares the popularity of each committee member to every non-member.

In this paper, we extend the notion of Gehrlein-stability in the unimodal setting [24] to the multimodal setting. Gehrlein-stability has been studied quite extensively for the committee selection problem in recent years [2,25,10,31,28]. It has been argued by Aziz et al. [2] that Gehrlein-stable committees are natural choice for shortlisting of candidates in situations that mirror multiwinner elections to avoid controversy surrounding inclusion of some candidate and exclusion of others as noted previously by [33,17]. Hence, there are good reasons to believe that a Gehrlein-stable committee for multimodal preferences will ably model scenarios described above. There are two notions of Gehrlein-stable committee in the unimodal setting, namely, *Strongly Gehrlein-stable committee*, and *Weakly Gehrlein-stable committee*, depending on margin of victory between two

[6] There are several other ways to submit a ballot.

candidates. A committee is *strongly* (*weakly*) *Gehrlein-stable*, if each committee member, v, is preferred by more than (at least) half of the voters over any non-committee member, u, in the pairwise election between u and v. The problem of finding strongly (weakly) Gehrlein stable committee is called Strongly (Weakly) Gehrlein Stable Committee Selection or S(W)GSCS in short. In the multimodal setting, we extend these definitions in a way that will capture our goal that the winning committee is "great across several attributes". Naturally, there may be several ways of achieving this. Chen et al. [9] undertakes one such study in the context of the stable matching problem, where instead of a committee, the goal is to pick a matching that satisfied some notion of stability in multiple preference profiles. In this paper, we use similar ideas to motivate notions of desirable solutions for the Multimodal Committee Selection problem that we believe are compelling, namely: *global stability*, *individual stability*, and *pairwise stability*, where each notion may be further refined in terms of strong or weak stability.

Our Model. Formally stated, for a positive integer ℓ, a *multimodal election* $\mathscr{E}$ with ℓ attributes (called *layers*) is defined by a set $\mathcal{C}$ of candidates, and a multi-set of ℓ preference profiles $(\mathcal{L}_i)_{i\in[\ell]}$, where each $\mathcal{L}_i$ is a multi-set of strict rankings of the candidate set, representing the voters (model is oblivious to voter set). The input instance of the Multimodal Committee Selection problem is a multimodal election $\mathscr{E} = (\mathcal{C}, (\mathcal{L}_i)_{i\in[\ell]})$, and two integers $\alpha, k \geq 1$ where $\alpha \in [\ell]$.[7] The goal is to find a k-sized committee that satisfies certain stability criteria, defined below, in α layers. We say that

- a committee S is *globally-strongly (weakly) stable* if there exist α layers in which S is strongly (weakly) Gehrlein-stable.
- a committee S is *individually-strongly (weakly) stable* if for each (committee member) $c \in S$, there exist α layers in which c is preferred by more than (at least) half of the voters over every (non-committee member) $d \in \mathcal{C} \setminus S$ in the pairwise election between c and d. We say that these layers provide stability to the candidate c, and c is *individually-strongly (weakly) stable* in these layers.
- a committee S is *pairwise-strongly (weakly) stable* if for each pair of candidates $\{c, d\} \subseteq \mathcal{C}$, where $c \in S$ and $d \in \mathcal{C} \setminus S$, there exist α layers in which c is preferred by more than (at least) half of the voters in the pairwise election between c and d. We say that these layers provide stability to the pair $\{c, d\}$, and the pair $\{c, d\}$ is *pairwise-strongly (weakly) stable* in these layers.

In our model, we do not assume that α is a function of ℓ. However, when there exists a relationship, we are able to exploit it (e.g., Theorem 15). In fact, it is very well possible that ℓ is large and $\alpha = 1$, for example, suppose the committee to be selected is a panel of experts to adjudicate fellowships. Each member of the panel is an expert in one field and while the panel size is k, there are some ℓ different subjects under consideration. In situations like these $\alpha = 1$.

We call a stable committee as a solution of the multimodal committee selection problem.

[7] For any $x \in \mathbb{N}$, $[x]$ denotes the set $\{1, 2, \ldots, x\}$.

Problem Names

We denote the problems of computing a globally-strongly (weakly) stable solution by G-SS (G-WS); an individually-strongly (weakly) stable solution by I-SS (I-WS); and a pairwise-strongly (weakly) stable solution by P-SS (P-WS). Additionally, for any $X \in \{G, I, P\}$, we will use X-YS to refer to both X-SS and X-WS.

For $X \in \{G, I, P\}$ and $Y \in \{\mathrm{S}, \mathrm{W}\}$, the formal definition of the problem is presented below.

X-YS

Input: A multimodal election $\mathscr{E} = (\mathcal{C}, (\mathcal{L}_i)_{i \in [\ell]})$, and two integers $\alpha, k \geq 1$, where $\alpha \in [\ell]$.
Question: Does there exist a committee of size k that is a solution for X-YS?

Remark 1. All of the definitions coincide with that of Srongly (Weakly) Gehrlein-stability when $\ell = \alpha = 1$.

Remark 2. The notion of strong and weak stability are equivalent for the odd number of voters.

Remark 3. A committee that is globally stable is also individually and pairwise stable; a committee which is individually stable is also pairwise stable.

Example 1. We explain our model using the following example containing 3 voters $\{v_1, v_2, v_3\}$, 4 layers $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4\}$, and 4 candidates $\{a, b, c, d\}$.

$$\begin{array}{llll}
 & v_1 & v_2 & v_3 \\
\mathcal{L}_1: & b \succ a \succ d \succ c; & a \succ b \succ d \succ c; & d \succ b \succ a \succ c \\
\mathcal{L}_2: & b \succ a \succ d \succ c; & a \succ d \succ c \succ b; & b \succ c \succ a \succ d \\
\mathcal{L}_3: & c \succ b \succ d \succ a; & c \succ a \succ d \succ b; & d \succ c \succ a \succ b \\
\mathcal{L}_4: & c \succ b \succ a \succ d; & d \succ c \succ b \succ a; & c \succ a \succ b \succ d
\end{array}$$

Let $\alpha = 2, k = 2$. Let $S = \{a, b\}$. In $\mathcal{L}_1$, v_1 and v_2 prefers a and b over c and d. Thus, S is strongly Gehrlein-stable in $\mathcal{L}_1$. In $\mathcal{L}_2$, v_1 and v_2 prefer a over c and d, and v_1 and v_3 prefer b over c and d. Thus, there exist 2 layers in which S is strongly Gehrlein-stable. Hence, S is globally-strongly stable. Next, let us consider a committee $S = \{b, c\}$. Note that S is not strongly Gehrlein-stable in any layer, thus, it is not a globally-strongly stable committee. However, b is more preferred than non-committee members a and d in layers $\mathcal{L}_1$ and $\mathcal{L}_2$, c is more preferred than a and d in the layers $\mathcal{L}_3$ and $\mathcal{L}_4$. Thus, b is individually-strongly stable in the layers $\mathcal{L}_1$ and $\mathcal{L}_2$, and c is individually-strongly stable in the layers $\mathcal{L}_3$ and $\mathcal{L}_4$. Hence, $S = \{b, c\}$ is individually-strongly stable. Let us consider a committee $S = \{b, d\}$. Note that S is neither globally-strongly stable nor individually-strongly stable as d is not more preferred than both a and c in any layer. However, d is more preferred than a in layers $\mathcal{L}_3$ and $\mathcal{L}_4$, and d is more preferred than c in layers $\mathcal{L}_1$ and $\mathcal{L}_2$. Furthermore, b prefers a and c both in $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence, $S = \{b, d\}$ is pairwise-strongly stable.

Differences between the notions. Note that for an instance of a Multimodal Committee Selection, it may be the case that it has no globally stable solution but has an individually stable solution. Moreover, it may also be the case that an instance may not have a globally stable or an individually stable solution but has a pairwise stable solution. We explain it using an example in the appendix and in Example 1.

Graph-theoretic Formulation. Similar to Gehrlein-stable model, all the models of stability that we study for multimodal election can be transformed to graph-theoretic problems on directed graphs. Using each of ℓ preference profiles, we create ℓ directed graphs with $\mathcal{C}$ as the vertex set, where in the i^{th} *layer*, denoted by the directed graph $G_i = (\mathcal{C}, \mathcal{A}_i)$, there is an arc from vertex a to b in $\mathcal{A}_i$ if and only if in $\mathcal{L}_i$ the candidate[8] a is preferred by more than half of the voters over b in the pairwise election between a and b. These directed graphs are known as *majority graphs* in the literature [2].

Let $S \subseteq \mathcal{C}$. In the language of the majority graph, S is strongly Gehrlein-stable in the i^{th} layer if for every pair of vertices u, v such that $u \in S$ and $v \in \mathcal{C} \setminus S$, v is an out-neighbor of u in G_i, which demonstrates that u is preferred over v by more than half of the voters. The set S is weakly Gehrlein-stable in the i^{th} layer if for every pair of vertices u, v such that $u \in S$ and $v \in \mathcal{C} \setminus S$, v is not an in-neighbor of u in G_i (i.e., either (u, v) is an arc or there is no arc between u and v), which demonstrates that u is preferred over v by at least half of the voters. We say that for the committee S, the vertex $u \in S$ is individually-strongly stable in the i^{th} layer if every $v \in \mathcal{C} \setminus S$ is an out-neighbor of u in G_i, and is individually-weakly stable if every in-neighbor of u in G_i is in S. Analogously, for the set S, a pair of vertices $u \in S$ and $v \in \mathcal{C} \setminus S$ is pairwise-strongly stable in the i^{th} layer if v is an out-neighbor of u in G_i, and is pairwise-weakly stable if v is not an in-neighbor of u in G_i. Note that when the numbers of voters is odd, all the graphs are *tournaments* (a directed graph in which there is an arc between every pair of vertices) and strongly and weakly stable definitions coincides to be the same. We will use graph-theoretic formulation for deriving our results.

Our contributions. Due to Remark 1, and NP-hardness and W[1]-hardness of WGSCS[9] with respect to k [2,25], G-WS, I-WS, and P-WS are NP-hard and W[1]-hard with respect to k. We list our contributions here. The notation $\mathcal{O}^\star(f(k))$ suppresses factors polynomial in input size.

- G-SS can be solved in polynomial time and G-WS in $\mathcal{O}^\star(1.2207^n)$ time for constant α, where n is the number of vertices in each layer. Furthermore, when all the layers are tournament graphs, G-WS can be solved in polynomial time due to Remark 2. Both the results are due to the reduction to unimodal case.
- I-WS is NP-hard and W[1]-hard with respect to k even when all the graphs are tournaments and $\alpha = 1$. This result is in contrast to unimodal case. Furthermore, it remains intractable even for transitive tournaments (an acyclic

[8] In the graph-theoretic formulation, we will refer to the candidates as vertices.

[9] GSCS is used in [25] as they only considered weak stability notion

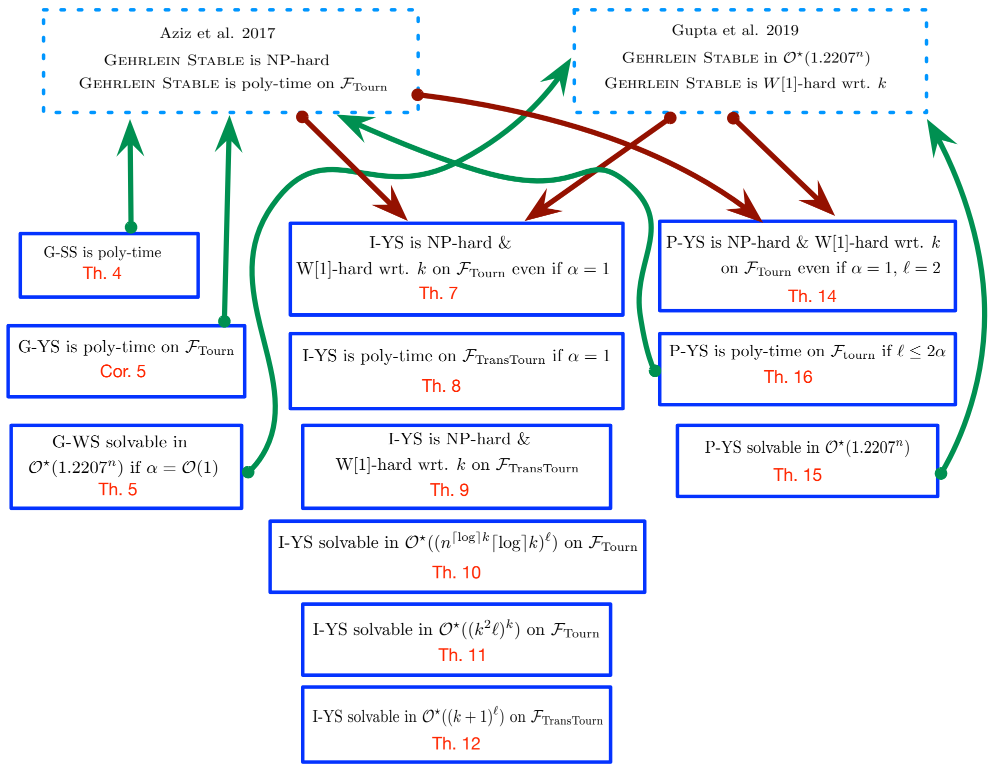

Fig. 1. Our Contributions. The **green arrows** to the dashed boxes represent reductions that led to an algorithm, and the **red arrows** from the dashed boxes represent reductions that led to a hardness result.

tournament), but in this reduction α is not constant. When all the graphs are transitive tournaments and $\alpha = 1$, it is solvable in polynomial time.

- When all the graphs are tournaments, we give following algorithms for I-WS:
  - solvable in $\mathcal{O}^\star((n^{\lceil \log k \rceil} \lceil \log k \rceil)^\ell)$ time. Thus, for $\ell \le \log n$, the problem is unlikely to be NP-hard unless NP $\subseteq$ QP[10].
  - solvable in $\mathcal{O}^\star((k\ell)^k)$ time.
- When all the graphs are transitive tournaments, I-WS can be solved in $\mathcal{O}^\star((k+1)^\ell)$ time.
- P-WS is NP-hard and W[1]-hard with respect to k even when all the graphs are tournaments, $\ell = 2$, and $\alpha = 1$. However, it can be solved in polynomial time when all the graphs are tournaments and $\ell < 2\alpha$.
- P-YS can be solved in $\mathcal{O}^\star(1.2207^n)$ time.

Figure 1 explain the interplay of results and their relations with each other. We skip the motivation for the considered parameters here as it is same as in [26].

[10] Here QP denotes the complexity class quasi-polynomial

Next, we highlight the significance of our study on tournaments and transitive tournaments.

Restrictions on layers. Aziz et al. [2] show that in the unimodal case a Gehrlein-stable committee can be found in polynomial time when the number of voters is odd, which corresponds to the case when majority graph is a tournament. Moreover, they also show that additionally if the preference lists satisfy *single-peaked* or *single-crossing* properties, then the corresponding majority graph is a transitive tournament (the graph can be a tournament or transitive tournament even in some other scenarios). Such domain restrictions are also studied by [26]. This motivates us to study the MULTIMODAL COMMITTEE SELECTION problem when each layer is a tournament or a transitive tournament.

Related works. Jain and Talmon [26] studied committee selection under some *mulimodal voting rules.* They discussed the significance of this problem, proposed generalisation of known committee scoring rules [20] to the multimodal setting, and studied computational and parameterized complexity of the multimodal variants of k-Borda and Chamberlin-Courant (CC). Chen et al. [9] gave similar definitions for stability for matching with multimodal preferences. Steindl and Zehavi [34] studied pareto optimal allocations of indivisible goods with multimodal preferences. Boehmer and Niedermeier [4] also highlighted the importance of multimodal preferences. There has been many works on multiwinner elections where the preference profile is attribute based [1,6,8,32,14,29,27] .

For the committee selection problem, extensive research has been conducted to study voting rules and their stability in the context of selecting a committee [10,17,28,33,19]. We refer to some surveys for application of parameterized complexity in social choice theory [5,18,15].

2 Preliminaries

Standard definitions and notations of graph theory in [13] apply. Let $G = (V, A)$ be a directed graph. For a vertex $v \in V(G)$, $N^-(v) = \{u : (u, v) \in A(G)\}$ denote the in-neighborhood of the vertex v. For a subset $X \subseteq V(G)$, $N^-(X)$ is the set of all in-neighbors of the vertices in X. $\mathscr{F}_{\mathsf{Tourn}}$ and $\mathscr{F}_{\mathsf{TransTourn}}$ denote the sets of graphs that contain tournaments and transitive tournaments, respectively. Unless explicitly specified, for two vertices u and v, both (u, v) and (v, u) are not arcs together in a directed graph. We use n to denote the number of vertices in a graph. *Topological ordering* of a directed graph G is a linear ordering of $V(G)$ where u precedes v for each arc (u, v). From the stability definitions, we have the following.

Proposition 1. *For any $X \in \{G, I, P\}$, an X-strongly stable and X-weakly stable solution are the same on $\mathscr{F}_{\mathsf{Tourn}}$.*

The following will be used for some of our algorithms.

Proposition 2. [25, Theorem 3] WGSCS *can be solved in time $\mathcal{O}^\star(1.2207^n)$ where n is the number vertices in the majority graph.*

We wish to point out that all our hardness reductions produce an instance where each layer is a directed graph (with arcs in only one direction). Thus, due to the following theorem, we can construct an election as well.

Proposition 3. [30] *Given a directed graph, there exists a corresponding election with size polynomial in the size of the given graph.*

Parameterized Complexity. Here, each problem instance is associated with an integer, k called *parameter*. A problem is said to be *fixed-parameter tractable* (FPT) with parameter k if it can be solved in $f(k)n^{\mathcal{O}(1)}$ time for some computable function f, where n is the input size. W-hardness captures the parameterized intractability with respect to a parameter. We refer the reader to [11,16,22] for further details.

When referring to a solution that is strongly(weakly) Gehrlein-stable, we may just say strongly(weakly) stable.

3 Global Stability

Here, we present results pertaining to G-YS, $Y \in \{\mathrm{S}, \mathrm{W}\}$.
Global-Strong Stability. Note that since each layer has a unique strongly stable committee [3, Theorem 1][11], we can "guess" a layer in which the solution is stable and then compute the strongly stable committee in that layer. Next, we verify if there are $\alpha - 1$ other layers in which that committee is also stable. Thus, we have the following:

Theorem 1. *(♠)*[12] G-SS *is solvable in polynomial time.*

Remark 4. Note that the strongly stable committee is unique in a unimodal election [3, Theorem 1], however the same is not true for a multimodal election as seen by the following example: Consider two majority graphs G_1 and G_2 on the vertex set $\{u, v, w\}$. Let arc sets be $E(G_1) = \{(u, v), (v, w), (u, w)\}$ and $E(G_2) = \{(v, u), (v, w), (w, u)\}$. For $k = 2$ and $\alpha = 1$, $S_1 = \{u, v\}$ and $S_2 = \{v, w\}$, both are globally-strongly stable.

Remark 5. Unlike strong stability, weak stable committee need not be unique, even for a unimodal election.

Global-Weak Stability. Next, we study parameterized complexity and a tractable case of G-WS. The hardness results, NP-hardness and W[1]-hardness with respect to k, which follows from intractability of WGSCS [2,25], motivates us to study parameterization with respect to n. In the following discussions, we will adopt the following terminology about G-WS: For an instance $((G_i)_{i\in[\ell]}, \alpha, k)$ and a subset of vertices S, we say that the i^{th} layer *provides*

[11] In [2], the term "strict" is used instead of "strong" (Def. 1 and first para in Sec 5 of [2])
[12] The proofs marked by ♠ can be found in supplementary.

stability to S if for any $u \in S$ and any $v \in V(G) \setminus S$ there is no arc (v, u) in the graph G_i.

The following algorithm works on the same idea as Theorem 1, the difference being that in light of Remark 5, it may not be sufficient to guess one layer and proceed as in Theorem 1. Here, we would need to know the solution in the layer it is stable and then verify if there are other layers which provide stability to the committee is also stable. An exhaustive search of such a committee would look through $\binom{n}{k}$ possibilities. Instead, if we guess the α layers, then we would have to find a solution that is weakly stable in those layers only, captured by a graph which is the union of the arc set in each of those layers. This gives an improvement in time if α is a constant.

Theorem 2. *(♠) G-WS can be solved in $\mathcal{O}^\star(1.2207^n)$ time, for $\alpha = \mathcal{O}(1)$.*

Proposition 1 and Theorem 1 imply the following result.

Corollary 1. *G-YS is solvable in polynomial time on $\mathscr{F}_{\mathsf{Tourn}}$.*

4 Individual Stability

In this section, we will discuss results pertaining to I-YS, where $Y \in \{\mathrm{S}, \mathrm{W}\}$.

4.1 Intractable Cases

We begin with an intractability result for tournaments. This is a sharp contrast to the unimodal case which is polynomial time solvable for $\mathscr{F}_{\mathsf{Tourn}}$.

Theorem 3. *I-YS is NP-hard and $\mathsf{W}[1]$-hard with respect to k on $\mathscr{F}_{\mathsf{Tourn}}$ even when $\alpha = 1$.*

Proof. We give a parameter-preserving reduction from the WGSCS problem, which is known to be $\mathsf{W}[1]$-hard with respect to k [25], to I-WS. Moreover, this being a polynomial time reduction will also prove that I-WS is NP-hard. Let $\mathcal{I} = (G, k')$ be an instance of WGSCS, where G is not a tournament; otherwise the instance is polynomial-time solvable. Let Z denote the set of vertices in G whose total degree (sum of in-degree and out-degree) is less than $n - 1$.

Construction. We will construct an instance of I-WS with $|Z|$ layers and $\alpha = 1$. For each vertex $u \in Z$, we create a graph G_u as follows. Initialize $G_u = G$, i.e., every arc in G also exists in each layer of $(G_u)_{u \in Z}$. Consider a vertex v which is neither an in-neighbor nor an out-neighbor of u. Then, we add an arc from u to v in G_u. We make G_u a tournament by adding the remaining missing arcs in an arbitrary direction. Clearly, this construction takes polynomial time.

Due to Proposition 3, G_u is a majority graph for an appropriately defined election. Note that the vertex set of G_u is same for each $u \in Z$. Hence, $\mathcal{J} = ((G_u)_{u \in Z}, \alpha = 1, k = k')$ is an instance of I-WS, where each directed graph G_u is a tournament. The next observations follow directly from the construction.

Observation 4 *Any vertex $u \in Z$ has the same set of in-neighbors in G and G_u.*

Observation 5 *Let $G' \in \{G_u : u \in Z\}$. Then, any vertex $v \in V(G) \setminus Z$ has the same in-neighbors in G and G'.*

The following shows the correctness of the reduction.

Lemma 1. *(♠) S is a solution for* WGSCS *in $\mathcal{I}$ iff S is a solution for* I-WS *in $\mathcal{J}$.*

Since the constructed graph is a tournament, we can conclude the intractability of I-YS. ☐

In contrast with the above intractability result, we note that when the layers are transitive tournaments and $\alpha = 1$, we have a tractable case for I-YS.

Theorem 6. I-YS *is solvable in polynomial time on $\mathscr{F}_{\mathsf{TransTourn}}$ if $\alpha = 1$; and a solution always exists.*

Proof. Let $\mathcal{I} = ((G_i)_{i \in [\ell]}, \alpha, k)$ be an instance of I-YS. Since each layer is a transitive tournament, we may assume that the vertices in the i^{th} layer, for $i \in [\ell]$, are ordered in terms of the topological ordering in G_i. Thus, We can find a solution by picking the first k vertices from G_1.

Unsurprisingly perhaps, for any arbitrary $\alpha > 1$ the problem is again intractable.

Theorem 7. I-YS *is* NP*-hard and* W[1]*-hard with respect to k on $\mathscr{F}_{\mathsf{TransTourn}}$.*

Proof. We prove this hardness result by showing a polynomial time reduction from Clique on regular graphs, in which given a regular undirected graph G and an integer k, the goal is to decide if there is a subset $S \subseteq V(G)$ of size k such that for every pair of vertices $u, v \in S$, uv is an edge in G. The Clique problem for regular graphs is NP-hard and W[1]-hard with respect to k [23,7]. Due to Proposition 1, we use I-SS in the rest of the proof.

We explain the construction along with the intuition behind the gadget. The precise construction of the transitive tournaments is in the **black** box below.

Construction: Let $(G, \tilde{k})$ be an instance of Clique, where degree of every vertex in G is d. For the ease of explanation, we assume that d is even. Let n and m denote the number of vertices and edges in G, respectively. We construct an instance of I-SS as follows: For every edge $e = uv$ in G, we have two directed graphs, say $\mathcal{M}_{e_u}$ and $\mathcal{M}_{e_v}$. For every edge $e \in E(G)$ and every vertex $u \in V(G)$, we add vertices u and e in every directed graph. We call these vertices as the *real vertices* of the directed graphs. For every directed graph $\mathcal{M}$ (constructed so far), we also add a set of dummy vertices, denoted by $D_{\mathcal{M}} = \{d^1_{\mathcal{M}}, \ldots, d^j_{\mathcal{M}}\}$, where the value of j will be specified later (at the end of the construction). We call these vertices as *dummy vertices* of the directed graphs.

The purpose of adding the dummy vertices is that a real vertex e corresponding to the edge $e \in E(G)$ should get the stability only from the corresponding directed graphs, and the real vertex u corresponding to the vertex $u \in V(G)$ should get stability only from the directed graphs $\mathcal{M}_{e_u}$, where e is an edge incident to u.

Since every transitive tournament has a unique topological ordering, we explain this ordering of vertices in every directed graph. Then, the arc set is self-explanatory. For the directed graph $\mathcal{M}_{e_u}$, the ordering is $(u, e, D_{\mathcal{M}_{e_u}}, \langle\text{remaining vertices}\rangle)$. The notation $\langle\cdot\rangle$ denote that the vertices in this set can be ordered in any arbitrary order. Intuitively, the goal is that if the vertex e is in the committee, then to provide it stability in the required number of layers (the number of layers will be defined later), u and v must also be in the solution (i.e., a vertex corresponding to an edge of G pulls vertices that correspond to its endpoints in G in the committee).

Next, we want to prevent more than $\tilde{k}$ vertices in the committee corresponding to vertices in $V(G)$, so that these vertices corresponds to clique vertices. Towards this, for every vertex $u \in V(G)$, we add a set of $\tilde{k}^2 - 1$ vertices, denoted by $T_u = (t_u^1, \ldots, t_u^{\tilde{k}^2-1})$, in every directed graph. We call these vertices as *indicator vertices*. Let $\overleftarrow{T_u}$ denote the set of vertices in the reverse order of T_u, i.e., $\overleftarrow{T_u} = (t_u^{\tilde{k}^2-1}, \ldots, t_u^1)$. Let $E(u)$ denote the set of edges incident to u. Let $E_1(u)$ and $E_2(u)$ be two disjoint sbsets of $E(u)$, each of size $|E(u)|/2$. In the ordering of the vertices of the directed graph $\mathcal{M}_{e_u}$, where $e \in E_1(u)$, we add T_u in front of the ordering constructed above, i.e., the new ordering of $\mathcal{M}_{e_u}$ is $(T_u, u, e, D_{\mathcal{M}_{e_u}}, \langle\text{remaining vertices}\rangle)$. For $e \in E_2(u)$, the ordering of the vertices of $\mathcal{M}_{e_u}$ is $(u, \overleftarrow{T_u}, e, D_{\mathcal{M}_{e_u}}, \langle\text{remaining vertices}\rangle)$, i.e., $\overleftarrow{T_u}$ is after u.

Additionally, for every edge $e \in E(G)$, we add $d/2 - 1$ many *dummy layers*, $\mathcal{M}_{e^i}$, $i \in [d/2-1]$, in which e is the first vertex in the ordering. Next, to ensure that no other real vertex get stability from these dummy layers, for every $\mathcal{M}_{e^i}$, $i \in [d/2 - 1]$, we add a new set of j dummy vertices, denoted by $D_{\mathcal{M}_{e^i}}$. The ordering of the vertices in these directed graphs is $(e, \langle D_{\mathcal{M}_{e^i}}\rangle, \langle\text{remaining vertices}\rangle)$. Note that for every $i \in [d/2 - 1]$, $\mathcal{M}_{e^i}$ provide stability to vertex e as it does not have any in-neighbors in these directed graphs. Note that the number of layers in the constructed instance is $m(d/2 + 1)$.

Finally, we set $k = \tilde{k}^3 + \binom{\tilde{k}}{2}$, $\alpha = d/2 + 1$, and the value of j as k so that no dummy vertex can be part of the solution.

Precisely, the construction is as follows.

Construction of an instance in the proof of Theorem 7

- For every $u \in V(G)$ and $e \in E(G)$, we add vertices u and e to directed graphs.
- For every $e(= uv) \in E(G)$, we add $d/2 + 1$ directed graphs, $\mathcal{M}_{e_u}, \mathcal{M}_{e_v}, \mathcal{M}_{e^1}, \mathcal{M}_{e^2}, \ldots, \mathcal{M}_{e^{d/2-1}}$.
- For every directed graph $\mathcal{M}$, we add a set of $\tilde{k}^3 + \binom{\tilde{k}}{2}$ dummy vertices $D_{\mathcal{M}} = d^1_{\mathcal{M}}, \ldots, d^{\tilde{k}^3+\binom{\tilde{k}}{2}}_{\mathcal{M}}$.
- For every vertex $u \in V(G)$, we add a set of indicator vertices $T_u = \{t^1_u, \ldots, t^{\tilde{k}^2}_u\}$.
- To define the edge set of a directed graph, we define its topological ordering. Let $E(u)$ denote the set of edges incident to u, and $E_1(u)$ and $E_2(u)$ be two disjoint subsets of $E(u)$ such that size of both the sets is $|E(u)|/2$.
  - For every $e \in E_1(u)$, the ordering of vertices in $\mathcal{M}_e$ is $(T_u, u, e, \langle D_{\mathcal{M}_e} \rangle, \langle\text{remaining vertices}\rangle)$
  - For every $e \in E_2(u)$, the ordering of vertices in $\mathcal{M}_e$ is $(u, \overleftarrow{T_u}, e, \langle D_{\mathcal{M}_e} \rangle, \langle\text{remaining vertices}\rangle)$
- For every $i \in [d/2 - 1]$, the ordering of vertices in $\mathcal{M}_{e^i}$ is $(e, \langle D_{\mathcal{M}_{e^i}} \rangle, \langle\text{remaining vertices}\rangle)$.
- $k = \tilde{k}^3 + \binom{\tilde{k}}{2}$ and $\alpha = d/2 + 1$.

Let $\mathcal{Z} = \{e_u, e_v : e(= uv) \in E(G)\} \cup \{e^i : e \in E(G), i \in [d/2 - 1]\}$. Since the set of vertices is same in all the directed graphs, we denote it by $V_{\mathcal{M}}$.

Next, we prove the correctness in the following lemma.

Lemma 2. *$\mathcal{I}$ is a yes-instance of* Clique *iff $\mathcal{J}$ is a yes-instance of* I-SS.

Proof. In the forward direction, let S be a solution to $(G, \tilde{k})$. Let $S' = \{\{u, T_u\} \subseteq V_{\mathcal{M}} : u \in S\} \cup \{e \in V_{\mathcal{M}} : e \in E(G[S])\}$, i.e., S' contains real and indicator vertices corresponding to the vertices and edges in $G[S]$. We claim that S' is a solution for $((\mathcal{M}_\ell)_{\ell \in \mathcal{Z}}, \alpha, k)$. Since for every $u \in V(G)$, $|T_u| = \tilde{k}^2 - 1$, and S is a $\tilde{k}$-sized clique, we have that $|S'| = \tilde{k} + \tilde{k}(\tilde{k}^2 - 1) + \binom{\tilde{k}}{2} = k$. Next, we argue that S' is individually stable for $\alpha = d/2 + 1$. Note that there are $d/2$ directed graphs in which the vertex u corresponding to the vertex $u \in V(G)$ does not have any in-neighbor, and there are $d/2$ directed graphs in which the in-neigbor of u is T_u. Since if $u \in S'$, $T_u \subseteq S'$, we have that there are at least $d/2 + 1$ directed graphs that provides individual stability to the vertex $u \in S'$. Similarly, there are at least $d/2 + 1$ directed graphs that provides individual stability to every vertex in T_u, where $T_u \subseteq S'$. Next, we argue about the vertex $e \in S'$ corresponding to the edge $e(= uv) \in E(G)$. Note that there are $d/2 - 1$ directed graphs (in particular, $\mathcal{M}_{e^i}$, where $i \in [d/2 - 1]$) in which e does not have any in-neighbor. Furthermore, in the directed graph $\mathcal{M}_{e_u}$, the set of in-neighbors of e is $T_u \cup \{u\}$ which is a subset of S' as $u \in S$. Similarly, all the in-neighbors of e in $\mathcal{M}_{e_v}$ belong to S'. Thus, S' is individually stable for $\alpha = d/2 + 1$.

In the backward direction, let S be an individually stable committee for $((\mathcal{M}_\ell)_{\ell \in \mathcal{Z}}, \alpha, k)$. We observe some properties of the set S.

Claim 1 *(♠) S does not contain any dummy vertex.*

Claim 2 *(♠) If $u \in S$, then $T_u \subseteq S$.*

Claim 3 *(♠) If $|T_u \cap S| \neq \emptyset$, then $u \in S$.*

Claim 4 *(♠) If the vertex e corresponding to the edge $e(= uv) \in E(G)$ is in S, then the vertices $\{u, v\} \subseteq S$.*

Let $V^\star = \{v \in V(G) \colon v \in S\}$ and $E^\star = \{e \in E(G) \colon e \in S\}$.

Claim 5 *(♠) $|V^\star| = \tilde{k}$ and $|E^\star| = \binom{\tilde{k}}{2}$.*

Next, we argue that the vertices are consistent with the edges, i.e., if $uv \in E^\star$, then $\{u, v\} \subseteq V^\star$. This follows from Claim 4. Moreover, since $|V^\star| = \tilde{k}$ and $|E^\star| = \binom{\tilde{k}}{2}$, it follows that the graph $G^\star = (V^\star, E^\star)$ is a complete graph on the vertex set $V^\star$, and thus $V^\star$ is a clique of size $\tilde{k}$ in G.

This completes the proof of the theorem. □

4.2 Tractable cases

The intractability results of Theorems 3 and 7 notwithstanding, motivate us to look for parameters beyond α and k. Specifically, we look for combined parameters and in doing so we show that for $Y \in \{\mathrm{S}, \mathrm{W}\}$, I-YS is FPT parameterized by $k + \ell$. We note that the parameterized complexity with parameter ℓ eludes us. However, Theorem 8 implies that when $\ell \leq \log n$, we have an algorithm with running time $2^{\mathrm{poly}(\log n)}$. Thus, we cannot hope for an NP-hardness result when $\ell \leq \log n$, unless NP $\subseteq$ QP. Therefore, the complexity when $\ell > \log n$ remains unknown.

At the heart of the parameterized algorithm, Theorem 8, is the notion of an *out-dominating set*, defined as follows. For any graph $G = (V, A)$, a set $S \subseteq V(G)$ is called an out-dominating set if every vertex $v \in V \setminus S$ has an out-neighbor in the set S.

Before we present the algorithm we can explain the intuition as follows. Any solution S for I-YS can be viewed as $S = S_1 \cup \ldots \cup S_\ell$, where each S_i denotes the set of vertices (possibly empty) that receive individual stability from the layer i. (Clearly, every vertex in S must be in at least α different S_is.) Moreover, we know that in the graph G_i the in-neighbors of any vertex in S_i are also present in S_i. Thus, S_i can be viewed as the union of a set X_i and the set of its in-neighbors in G_i, i.e., $S_i = X_i \cup N^-_{G_i}(X_i)$. The set X_i here is the out-dominating set of the subgraph induced by S_i in G_i, denoted by T_i. While we do not know the set S_i, we know that its size is at most k. Hence, the subgraph $T_i = G_i[S_i]$ has at most k vertices and has an out-dominating set of size at most $\lceil \log k \rceil$, due to Lemma 3. This allows us to enumerate all possible subsets of size $\lceil \log k \rceil$ and from that find its in-neighborhood. This process allows us to find X_i, $N^-(X_i)$, and thus S_i for each $i \in [\ell]$, and from there the set S.

Lemma 3. *A tournament $G = (V, A)$ has an out-dominating set of size at most $\lceil \log |V| \rceil$. Additionally, if G is a transitive tournament, then G has a unique out-dominating set of size one.*

Proof. From the definition of an out-dominating set we know that $V = X \uplus N^-(X)$, that is X and the set of its in-neighbors partition the vertex set of G.

Next, we show a counting argument using the Handshaking Lemma, which says that for a directed graph, $\sum_{v \in V} \delta^-(v) = \sum_{v \in V} \delta^+(v)$. We first show that every tournament (and thus G) has a vertex of in-degree at least $(n-1)/2$. Suppose not, then $\sum_{v \in V} \delta^-(v) < n(n-1)/2$ and $\sum_{v \in V} \delta^+(v) > n(n-1)/2$ as the total degree is $n(n-1)/2$. Since $\sum_{v \in V} \delta^-(v) = \sum_{v \in V} \delta^+(v)$, we have reached a contradiction.

Using the vertex with in-degree at least $\lceil (n-1)/2 \rceil$ we will recursively create an out-dominating set of size at most $\lceil \log n \rceil$. Let v be a vertex such that its in-degree is at least $\lceil (n-1)/2 \rceil$ in G. Then, set $X = \{v\}$ and $V = V \setminus N^-[v]$. This results in a tournament with a smaller vertex set. We recurse until the graph is empty. Since each time we take away a set of size at least $n/2$, this process can only go on for at most $\lceil \log n \rceil$ steps. Consequently, at the end we have a set X that has size at most $\lceil \log n \rceil$. The construction ensures that X is indeed an out-dominating set. ◻

Theorem 8. I-YS *is solvable in time $\mathcal{O}^\star((n^{\lceil \log k \rceil} \lceil \log k \rceil)^\ell)$ on $\mathscr{F}_{\mathsf{Tourn}}$.*

Proof. Let $\mathcal{I} = ((G_i)_{i \in [\ell]}, \alpha, k)$ be an instance of I-YS. For each $i \in [\ell]$, our algorithm guesses a vertex subset of size at most $\lceil \log k \rceil$ in G_i and finds its in-neighborhood set in G_i. The union of these two sets is denoted by Y_i. If $N_i^-(Y_i) \setminus Y_i \neq \emptyset$, then we set $Y_i = \emptyset$. Else, the algorithm checks if $\cup_{i \in [\ell]} Y_i$ is a solution for $\mathcal{I}$. If the algorithm fails to find a subset of vertices that is a solution for $\mathcal{I}$, then it returns "no".

Correctness. Any solution returned by the algorithm will quite obviously be a solution for $\mathcal{I}$. Thus, we only need to prove the other direction. Suppose that $\mathcal{I}$ is a *y*es-instance and S is a solution. We may view S as a union of ℓ (possibly empty) sets S_i where S_i contains the vertices of G_i that are stable in the layer i, i.e., all those vertices whose in-neighbors in G_i are also in S_i. For each $i \in [\ell]$, we consider the induced subgraph $T_i = G_i[S_i]$, which is the tournament induced by the vertices in S_i. For each $i \in [\ell]$, let X_i denote an out-dominating set of the graph T_i. Due to Lemma 3, $|X_i| \leq \lceil \log k \rceil$ since $|S_i| \leq k$; and $S_i = X_i \cup N_i^-(X_i)$, where $N_i^-(X_i)$ denotes the set of in-neighbors of X_i in G_i.

Our algorithm basically tries to generate the set X_i by trying all possible subsets of size at most $\lceil \log k \rceil$, and from that construct the set S_i. Thus, for some choice of Y_i we will have $Y_i = S_i$ for each $i \in [\ell]$ and then the algorithm will return the solution S. Suppose that $\cup_{i \in [\ell]} Y_i$ is a set returned by the algorithm, then by the last check we know that it is also a solution for the instance $\mathcal{I}$ of I-YS.

Time complexity. This results in an algorithm that has to verify at most $\left(\sum_{0 \leq i \leq \lceil \log k \rceil} \binom{n}{i}\right)^\ell$ different subsets of vertices since in any layer there are

$\sum_{0 \le i \le \lceil \log k \rceil} \binom{n}{i}$ different subsets of size at most $\lceil \log k \rceil$. The last verification step can be carried out in $\mathcal{O}(k\ell)$ steps by checking for each vertex in $\cup_i Y_i$ if there are α layers in which it is stable. $\square$

Next, we discuss an FPT algorithm for the parameter $k + \ell$. We begin with the following result that may be of independent interest.

Lemma 4. *(♠) In any tournament there are at most $2k + 1$ vertices with in-degree at most k.*

The next result is inspired by the above lemma as there are only $\mathcal{O}(k\ell)$ vertices that can be part of solution, $\mathcal{O}(k)$ from each layer.

Theorem 9. *(♠)* I-YS *is solvable in time $\mathcal{O}^\star((k\ell)^k)$ on $\mathscr{F}_{\mathsf{Tourn}}$.*

Remark 6. Comparing Theorem 8 vs Theorem 9. Note that neither algorithm subsumes the other. Each works better than the other in certain situations as described below

- For a constant value of k, Theorem 9 gives a polynomial time algorithm while Theorem 8 gives an $n^{\mathcal{O}(\ell)}$ time algorithm, (i.e., it does not run in polynomial time if ℓ is not a constant.)
- For a constant value of ℓ, Theorem 9 gives an FPT-algorithm with respect to k (i.e., it runs in polynomial time if k is also a constant), while Theorem 8 gives a quasi-polynomial time algorithm.

Notwithstanding the hardness of Theorem 7 on transitive tournaments, we note that the problem does admit polynomial time algorithm if the total number of layers is a constant, which is an improvement over the running times given by Theorems 8 and 9.

Theorem 10. *(♠)*I-YS *is solvable in $\mathcal{O}^\star((k+1)^\ell)$ time on $\mathscr{F}_{\mathsf{TransTourn}}$.*

Due to Theorem 10, we have the following.

Corollary 11 *(♠)* I-YS *is solvable in polynomial time on $\mathscr{F}_{\mathsf{TransTourn}}$ if $\ell = \mathcal{O}(\log_k n)$.*

Theorem 12. *(♠)* I-YS *is solvable in polynomial time on $\mathscr{F}_{\mathsf{TransTourn}}$ if $\ell = \alpha$.*

5 Pairwise Stability

In this section, we will discuss results pertaining to P-YS, where $Y \in \{\mathrm{S}, \mathrm{W}\}$. Note that for $\ell = 1$, P-YS can be solved in polynomial time on $\mathscr{F}_{\mathsf{Tourn}}$, however, for $\ell = 2$, we have the following intractability result.

Theorem 13. P-YS *is* NP*-hard and* W[1]*-hard with respect to k on $\mathscr{F}_{\mathsf{Tourn}}$ even when $\alpha = 1$ and $\ell = 2$.*

Proof. We give a reduction from an instance of WGSCS. Since WGSCS is W[1]-hard with respect to parameter k [25], this will prove that P-YS is also W[1]-hard with respect to k. Let (G, k') be an instance of WGSCS. We will create an instance of P-YS with two layers G_1 and G_2. Initialize $G_1 = G_2 = G$. Next, for every pair of vertices $\{u, v\}$ that do not have an arc between them in G, we add the arc (u, v) in G_1, and add the arc (v, u) in G_2. We define $\mathcal{J} = (G_1, G_2, \alpha = 1, k = k')$ to be an instance of P-YS. Note that G_1 and G_2 both are tournaments.

Since we can construct G_1 and G_2 in polynomial time, the following result proves the theorem.

Lemma 5. *(♠) S is solution for $\mathcal{I}$ iff S is a solution for $\mathcal{J}$.*

This completes the proof. □

The next result pertains to the parameterized complexity of P-YS with respect to n. We prove it by showing reductions to WGSCS.

Theorem 14. *(♠)* P-YS *is solvable in time in $\mathcal{O}^\star(1.2207^n)$.*

By focusing our attention towards structural parameters pertaining to the layers in the instance of P-YS, we obtain the following result.

Theorem 15. *(♠)* P-YS *is solvable in polynomial time on $\mathscr{F}_{\mathsf{Tourn}}$ if $\ell < 2\alpha$.*

We conclude our discussions with the following result about weak stability that follows due to the relationship between I-WS and P-WS, and Theorem 6.

Corollary 16 P-WS *is solvable in polynomial time on $\mathscr{F}_{\mathsf{TransTourn}}$ if $\alpha = 1$.*

6 Conclusion

We extend the study of stable committee to the multimodal elections. In fact, in [26], the authors considered the same set of voters and candidates across the layers. We generalise this to the scenario, where voters need not be the same across the layers, and justified this model in Introduction. We defined three notions of stability and studied their computational and parameterized complexity.

The following questions elude us so far for transitive tournaments: (i) the computational complexity of I-YS for constant $\alpha > 1$, (ii) the parameterized complexity of I-YS with parameter ℓ, (iii) the computational complexity of P-YS.

Jain and Talmon [26] initiated the study of scoring rules for multimodal multiwinner election. We believe that it would be interesting to extend the notion of stability given by Darmann [12] to multimodal preferences. In general, it would be interesting to extend the extensive study of multiwinner election for unimodal case to multimodal preferences.

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A Appendix

Proof (Theorem 1). Let $\mathcal{I} = ((G_i)_{i\in[\ell]}, \alpha, k)$ be an instance of G-SS. Let S denote a solution. We first guess a layer $i \in [\ell]$ in which S is strongly stable. Next, we find a strongly stable committee in i^{th} layer using the known polynomial time algorithm for finding strongly Gehrlein-stable committee [3, Theorem 1]. If there exist α many layers in $\mathcal{I}$ in which S is strongly stable, then we return "yes", otherwise we return "no". Next, we prove the correctness of the algorithm.

Suppose that S is a solution for G-SS in $\mathcal{I}$. Let $j \in [\ell]$ be a layer in which S is strongly stable. Note that in our algorithm, when we guess the value j, the algorithm will find the set S because a strongly stable solution is unique [3]. Clearly, by the definition of S, there exist α layers in which S is strongly stable, and so our algorithm will return "yes".

The other direction is trivial since we return "yes" only if there exist α layers in which S is strongly stable. $\square$

Proof (Theorem 2). Suppose that S is a solution to the problem. We start by guessing the α layers that give stability to S. Let $Z = \{G_1, \dots, G_\alpha\}$ denote the majority graphs in these α layers. This step takes $\binom{\ell}{\alpha}$ amount of time. Next, we create another graph $G' = (\mathcal{C}, \mathcal{A}')$, with the same set of vertices as any G_i, but the arc set $\mathcal{A}'$ is the union of the arcs in the graphs in $G_i \in Z$. We find a weakly stable committee S of size k in G' or return no solution exists in time $\mathcal{O}^\star(1.2207^n)$[25]. Since α is constant, the above algorithm takes time $\mathcal{O}^\star(1.2207^n)$. Next, we argue the correctness.

Correctness. We will argue that the above algorithm correctly finds a solution if one exists. We show that a committee S is a solution for G-WS iff S is a weakly stable committee in G'. We begin by showing that S (the output of the above algorithm) is a solution for G-WS. Let G be an arbitrarily chosen graph from the set Z. We claim that S is a weakly stable committee in the majority graph G, and thus S is weakly stable in every graph $G_i \in Z$. Since S is weakly stable in G', there is no arc (v, u) in $\mathcal{A}'$ where $u \in S$ and $v \in \mathcal{C} \setminus S$. We have $\mathcal{A}(G) \subseteq \mathcal{A}'$, so $\mathcal{A}$ does not contain an arc (v, u) such that $u \in S$ and $v \in \mathcal{C} \setminus S$. Thus, S is weakly stable in $G_i, i \in [\alpha]$; and consequently, S is a solution for G-WS.

For the other direction, let S be a solution of G-WS. Then, there exist α layers (i.e majority graphs) $(G_i)_{i\in[\ell]}$ such that S is weakly stable in G_i for each $i \in [\alpha]$. Since S is weakly stable in G_i, for each $i \in [\alpha]$, there is no arc (v, u) in $\mathcal{A}(G_i)$ such that $u \in S$ and $v \in \mathcal{C} \setminus S$. Given the definition, $\mathcal{A}' = \cup_{i\in[\alpha]}\mathcal{A}(G_i)$, there are no arcs (v, u) in $\mathcal{A}'$ such that $u \in S$ and $v \in \mathcal{C} \setminus S$. Hence, S is a weakly stable committee in G' as well. $\square$

Proof (Lemma 1). Suppose that S is a solution for the instance $\mathcal{I}$. Let v be a vertex in the solution S. Since S is a solution in G, no vertex outside S is an in-neighbor of v in G. If $v \in Z$, then using Observation 4, all in-neighbors of v in G_v are in S. If $v \in V(G) \setminus Z$, then using Observation 5, we know that no vertex outside S is an in-neighbor of v in G', for any graph $G' \in \{G_u : u \in Z\}$. Therefore, in both cases there is at least one layer in which no vertex outside the set S is an in-neighbor of v. Hence, S is also a solution for $\mathcal{J}$.

For the other direction, suppose that S is a solution for the instance $\mathcal{J}$. Note that $S \subseteq V(G)$ and thus, if it is also a solution of $\mathcal{I}$, then we are done. Suppose that S is not a solution for $\mathcal{I}$. It must be the case that for some vertex $v \in S$, there exists an in-neighbor $w \in V(G) \setminus S$. But from Observations 4 and 5, we know that in every graph G_u in $\mathcal{J}$, the in-neigbors of v in G are also its in-neighbors in G_u. Hence, in *every* majority graph G_u, vertex w which is outside S is an in-neighbor of v. This is a contradiction to S being a solution for $\mathcal{J}$. $\square$

Proof (Claim 1). Due to the construction, for every dummy vertex in $V_{\mathcal{M}}$, all but one directed graph has more than k in-neighbors, it follows that no dummy vertex is in S. $\square$

Proof (Claim 2). Let $E(u)$ be the set of edges incident to the corresponding vertex $u \in V(G)$. Note that in all the directed graphs except $\mathcal{M}_\ell$, where $\ell \in \{e_u : e \in E(u)\}$, u has more than k in-neighbors, so these layers cannot provide stability to u. Recall that u does not have any in-neighbor in $d/2$ many directed graphs among $\mathcal{M}_\ell$, and in $d/2$ many directed graphs the in-neighbor of u is only T_u whose size is $\tilde{k}^2 - 1 < k$. Since $\alpha = d/2 + 1$, we can infer that $T_u \subseteq S$. $\square$

Proof (Claim 3). Let $t \in T_u \cap S$. Note that in all but d directed graphs, t has more than k in-neighbors, so t does not receive stability from these layers. Out of these d directed graphs, in $d/2$ directed graph, the set of in-neighbors of t is a subset of T_u. However, in the remaining $d/2$ directed graphs, u is an in-neighbor of t. Since $\alpha = d/2 + 1$, we can infer that $u \in S$. $\square$

Proof (Claim 4). Recall that in all but $d/2 + 1$ directed graphs (in particular, $\mathcal{M}_{e^1}, \ldots, \mathcal{M}_{e^{d/2-1}}, \mathcal{M}_{e_u}, \mathcal{M}_{e_v}$), the vertex e has more than k in-neighbors. Therefore, if $e \in S$, then $\{u, v\} \subseteq S$. $\square$

Proof (Claim 5). Towards the contradiction, suppose that $|V^\star| = k^\star < \tilde{k}$, Due to Claims 2 and 3, S contains at most $k^\star(\tilde{k}^2 - 1)$ indicator vertices. Since $k = \tilde{k}^3 + \binom{\tilde{k}}{2}$, there must be more than $\binom{\tilde{k}}{2}$ vertices in S corresponding to edges in G. Due to Claim 4, if $e \in S$, where e is a vertex corresponding to the edge $e(= uv) \in E(G)$, then $\{u, v\} \subseteq S$. Since $|V^\star| = k^\star$, there are at most $\binom{k^\star}{2}$ vertices in S corresponding to edges in G. Thus, $|S| = k^\star \tilde{k}^2 + \binom{k^\star}{2} < k$, a contradiction. If $|V^\star| > \tilde{k}$. Then, due to Claim 2, $|S| \geq (\tilde{k} + 1)\tilde{k}^2 > k'$, a contradiction.

Hence, $|V^\star| = \tilde{k}$. Furthermore, due to Claims 2 and 3, the number of indicator vertices is $\tilde{k}(\tilde{k}^2 - 1)$. Due to Claim 1, the other vertices in S are corresponding to edges in G. Thus, $|E^\star| = \binom{\tilde{k}}{2}$. $\square$

Proof (Lemma 4). Let $G = (V, A)$ be a tournament and $V_k = \{v \in V \mid \delta^-_G(v) \leq k\}$. Then, $\sum_{v \in V_k} \delta^-_{G[V_k]}(v) = \binom{|V_k|}{2}$ since $G[V_k]$ is a tournament. Moreover, since $G[V_k]$ is a subgraph of G we have that $\sum_{v \in V_k} \delta^-_{G[V_k]}(v) \leq \sum_{v \in V_k} \delta^-_G(v) \leq k|V_k|$. The second inequality follows from the definition of V_k. Hence, we have the required bound from $\binom{|V_k|}{2} \leq k|V_k|$.

Proof (Theorem 9). Due to Lemma 4, every layer has at most $2k+1$ vertices to which it can provide stability. Since there are ℓ layers, there are at most $\ell(2k+1)$ vertices which can get stability from any layer. Let X be the set of these vertices. We try all possible subsets of X of size k and output the one which is individually-stable in at least α layers. If there is no such set, we return "no". The correctness follows from the fact that any individually-stable solution is a subset of X as all the vertices in $\mathcal{C} \setminus X$ has more than k in-neighbors in every layer. □

Proof (Theorem 10). For any layer, if it provides stability to any vertex, then the vertex must be among the first k vertices appearing in the unique topological ordering. So, we guess the last vertex in the topological ordering which gets stability from this layer. Thus, for each layer we have a choice of $k+1$ vertices ($+1$ due to the possibility that no vertex gets stability from the current layer). Thus, there are $(k+1)^{\ell}$ possible choices for a solution and each can be verified in polynomial time. □

Proof (Theorem 12). The proof follows from the observation that the given instance is a *y*es-instance iff the first k vertices in each layer are the same. So, we can take these first k vertices in the solution. □

Proof (Lemma 5). Let S be a weakly stable committee for the instance $\mathcal{I}$. Towards this, let u and v be two vertices such that $u \in S$ and $v \in V(G) \setminus S$. Since $\alpha = 1$, we need to show that there exists one layer in which (v, u) is not an arc. Since S is stable in G, (v, u) is not an arc in G. Therefore, from the construction, in G_1 and G_2 both, we have the arc (u, v). Hence, S is a solution for the instance $\mathcal{J}$.

For the other direction, let S be a solution for $\mathcal{J}$. Suppose that for contradiction there is an arc (v, u) in G such that $u \in S$ and $v \in V(G) \setminus S$. Since G is subgraph of both G_1 and G_2, the arc (v, u) must be in both G_1 and G_2, implying that S cannot be a solution for $\mathcal{J}$, a contradiction. □

Proof (Theorem 14). We first present an algorithm for P-SS followed by an algorithm for P-WS .

P-SS. Let $\mathcal{I} = ((G_i)_{i \in [\ell]}, \alpha, k)$ be an instance of P-SS. We construct a directed graph $G' = (V, \mathcal{A})$ as follows. Let $V = V(G_1)$. For a pair of vertices $\{u, v\} \subseteq V$, if there are strictly fewer than α layers in which (v, u) is an arc in $\mathcal{I}$, then we add the arc (u, v) to G'. (Note that in $\mathcal{I}$ if there are strictly fewer than α layers in which (v, u) is an arc, and also there are strictly fewer than α layers in which (u, v) is an arc, then both arcs (u, v) and (v, u) are in G'.)

The intuition behind this construction is the following: for a pair of vertices u and v, if there are fewer than α layers that contain the arc (v, u), then for v to be part of the solution for $\mathcal{I}$ would require that u is also part of the solution. This is equivalent to finding weakly Gehrlein-stable committee for $\mathcal{J} = (G', k)$ (The input instance of WGSCS contains a majority graph that contains arc in only one direction. However, we allow arc in both the directions here. We use the same term "weakly Gehrlein-stable committee" as here also the goal is to find a subset $S \subseteq V(G')$ whose in-neighborhood is in S) as for this case we add the arc (u, v) to G'. The following claim establishes the correctness of this approach.

Claim 6 *S is a solution for* P-SS *in $\mathcal{I}$ iff S is a solution for* WGSCS *in $\mathcal{J}$.*

Proof. Let S be a solution for the instance $\mathcal{I}$ but not for the instance $\mathcal{J}$. Then, there must exist a pair $\{u, v\} \subseteq V$ such that $v \in S$ and $u \in V \setminus S$ and G' has the arc (v, u). Hence, in $\mathcal{I}$ there must exist strictly fewer than α layers in which (u, v) is an arc, a contradiction to the definition of S.

Conversely, suppose that S is a solution for $\mathcal{J}$, but not a solution for $\mathcal{I}$. Then there exists a pair $\{u, v\}$, where $v \in S$ and $u \in V \setminus S$ such that there are strictly fewer than α layers in $\mathcal{I}$ which contain the arc (v, u). But then, G' must have the arc (u, v) which contradicts the definition of S. □

Since a weakly stable committee for the instance (G', k) can be found in time $\mathcal{O}^\star(1.2207^n)$, Proposition 2, our algorithm for P-SS runs in the stated time. The correctness of this algorithm follows from Claim 6. Note that the algorithm mentioned in Proposition 2 takes majority graph as input, however, it works for any directed graph. Thus, the result holds for P-SS.

Next, we give the algorithm for P-WS. We begin with the following lemma.

Lemma 6. *P-WS can be reduced to* WGSCS *in polynomial time.*

Proof. Let $\mathcal{I} = ((G_i)_{i\in[\ell]}, \alpha, k)$ denote an instance of P-WS. Let $V = V(G_1)$, and the instance of WGSCS is $\mathcal{J} = (G, k)$ where the directed graph $G = (V, \mathcal{A})$ is defined as follows. The arcs in $\mathcal{A}$ are supposed to capture the "pulling action" of vertices that end up in the solution. For example, an arc (v, u) in $\mathcal{A}$ represents the condition that if u is in the solution, then so is v. This will happen when there do not exist α layers in which the arc (v, u) does not exist. Formally stated, arc $(v, u) \in \mathcal{A}$ if and only if arc (v, u) exists in every α-sized subset of layers of $\mathcal{I}$. The following claim establishes the correctness of this approach.

Claim 7 *S is a solution for* P-WS *in $\mathcal{I}$ iff S is a solution for* WGSCS *in $\mathcal{J}$.*

Proof. Let S denote a weakly stable committee in G. Suppose that S is not a solution for $\mathcal{I}$. Then, it must be that there exists a pair $\{u, v\}$, where $u \in S$ and $v \in V \setminus S$ such that the arc (v, u) exists in every α-sized subset of layers of $\mathcal{I}$. Thus, due to the construction, the arc (v, u) must be in G; a contradicting to the definition of S.

For the other direction, let S be a solution for P-WS in $\mathcal{I}$. Therefore, for every pair of vertices $\{u, v\}$ such that $u \in S$ and $v \in V \setminus S$, there exist at least α layers in which (v, u) is not an arc. Therefore, the arc (v, u) also does not exist in G. Hence, S is weakly stable in G. □

This completes the proof. □

Due to Proposition 2 and Lemma 6, our algorithm for P-WS runs in the stated time.

Thus, the theorem is proved. □

Proof (Theorem 15). Let $\mathcal{I} = ((G_i)_{i\in[\ell]}, \alpha, k)$ be an instance of P-YS where each G_i is a tournament. Note that the above reduction in Theorem 14 to WGSCS can be applied to $\mathcal{I}$ resulting in an instance $\mathcal{J} = (G', k)$, where, we argue that, G' has at least one arc between every pair of vertices. Since $\ell < 2\alpha$, it follows that for any pair $\{u, v\} \subseteq V$ in $\mathcal{I}$, there cannot simultaneously exist α layers which contain the arc (u, v) as well as α layers which contain the arc (v, u). In other words, at most one of the arcs can only be present in at least α layers, and so we are ensured that G' will contain (at least) one arc between the vertices u and v.

Since WGSCS can be solved in polynomial time when the graph is a tournament [2,25] (the same algorithm can be used for semi-complete digraphs[13]), the above reduction would yield a polynomial time algorithm for P-YS. The correctness follows due to Lemma 6.

Thus, the theorem is proved. □

Relationship between Diverse Committee and Our Model. The problem of finding a globally-strongly stable committee on transitive tournaments can be reduce to the Diverse Committee problem in the polynomial time as follows. We consider the topological ordering of graphs in every layer. We divide each ordering into two candidate groups, consisting of top-k and bottom $m - k$ candidates in the ordering. For the set of top-k candidates we set the lower and upper bound both as k, and for the other set of candidates, we set both the bounds as 0. Now, we find a committee that satisfies at least α non-zero constraints. This is equivalent to G-SS on transitive tournaments as for the solution of G-SS on transitive tournaments is among top k-candidates in the topological ordering and this set should be the same for α many layers. However, this reduction cannot be generalised for arbitrary graphs in polynomial time. Since G-SS can be solved in polynomial time on tournaments, using known algorithm for Gehrlein-stability on tournaments, this reduction is not much useful for us, computationally.

[13] a class of graph in which there is at least one arc between every pair of vertices u and v