A Strongly-Uniform Slicewise Polynomial-Time Algorithm for the Embedded Planar Diameter Improvement Problem

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Abstract

In the EMBEDDED PLANAR DIAMETER IMPROVEMENT problem (EPDI) we are given as input a graph G embedded in the plane and a positive integer d. The goal is to determine whether one can add edges to G in such a way that the resulting graph is still planar but has diameter at most d. Using non-constructive techniques derived from Robertson and Seymour's graph minor theory, together with the effectivization by self-reduction technique introduced by Fellows and Langston, one can show that EPDI can be solved in time $f(d) \cdot |V(G)|^{O(1)}$ for some function $f(d)$. The caveat is that this algorithm is *not* strongly uniform in the sense that the function $f(d)$ is not known to be computable. On the other hand, even the problem of determining whether EPDI can be solved in time $f_1(d) \cdot |V(G)|^{f_2(d)}$ for *computable* functions f_1 and f_2 has been open for more than two decades [Cohen at. al. Journal of Computer and System Sciences, 2017]. In this work we settle this problem by showing that EPDI can be solved in time $f(d) \cdot |V(G)|^{O(d)}$ for some computable function f. Our techniques can also be used to show that the EMBEDDED k-OUTERPLANAR DIAMETER IMPROVEMENT problem (k-EOPDI), a variant of EPDI where the graph G' is required to be k-outerplanar instead of planar, can be solved in time $f(d) \cdot |V(G)|^{O(k)}$ for some computable function f. This shows that for each fixed k, the problem k-EOPDI is strongly uniformly fixed parameter tractable with respect to the diameter parameter d.

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1 Introduction

In the EMBEDDED PLANAR DIAMETER IMPROVEMENT problem (EPDI), we are given a plane graph G and a positive integer d, and the goal is to determine whether G has a plane supergraph G' of diameter at most d. The set of YES instances of EPDI is closed under minors. In other words, if G has a plane supergraph of diameter at most d, then any minor H of G also has such a supergraph. Therefore, using non-constructive arguments from Robertson and Seymour's graph minor theory [14, 15] in conjunction with the fact planar graphs of constant diameter have constant treewidth, one can show that for each fixed d, there exists an algorithm $\mathfrak{A}_d$ which determines in linear time whether a given G has diameter at most d. The caveat is that the non-constructive techniques mentioned above provide us with no clue about what the algorithm

$\mathfrak{A}_d$ actually is. This problem can be partially remedied using a technique called effectivization by self-reduction introduced by Fellows and Langston [10, 8]. Using this technique one can show that for some function $f : \mathbb{N} \rightarrow \mathbb{N}$, there exists a single algorithm $\mathfrak{A}$ which takes a plane graph G and a positive integer d as input, and determines in time $f(d) \cdot |V(G)|^{O(1)}$ whether G has a plane supergraph of diameter at most d. Nevertheless, the function $f : \mathbb{N} \rightarrow \mathbb{N}$ bounding the influence of the parameter d in the running time of the algorithm mentioned above is not known to be computable. The problem of determining whether EPDI admits an algorithm running in time $f(d) \cdot |V(G)|^{O(1)}$ for some *computable* function f is a notorious and long-standing open problem in parameterized complexity theory [8, 9, 5]. Interestingly even the problem of determining whether EPDI can be solved in time $f_1(d) \cdot |V(G)|^{f_2(d)}$ for *computable* functions f_1 and f_2 has remained open until now [4]. In this work we settle this latter problem by showing that EPDI can be solved in time $2^{d^{O(d)}} \cdot |V(G)|^{O(d)}$. The problem of determining whether EPDI can be solved in time $f(d) \cdot |V(G)|^{O(1)}$ for some *computable* function $f : \mathbb{N} \rightarrow \mathbb{N}$ remains widely open.

A graph is 1-outerplanar if it can be embedded in the plane in such a way that every vertex lies in the outer face of the embedding. A graph is k-outerplanar if it can be embedded in the plane in such a way that after deleting all vertices in the outer face, the remaining graph is $(k-1)$-outerplanar. In [4] Cohen et al. have considered the k-OUTERPLANAR DIAMETER IMPROVEMENT problem (k-OPDI), a variant of the PDI problem in which the target supergraph is required to be k-outerplanar instead of planar. In particular, they have shown that the 1-OPDI problem can be solved in polynomial time. The complexity of the k-OPDI problem with respect to explicit algorithms was left as an open problem for $k \geq 2$. By adapting our algorithm for the EPDI problem we are able to show that when the input graph is given together with an embedding that must be preserved, then the resulting problem, the k-EOPDI problem, can be solved in time $2^{d^{O(d)}} \cdot |V(G)|^{O(k)}$ for each fixed k. In other words, this problem is strongly uniformly fixed parameter tractable with respect to the diameter parameter for each fixed value of outerplanarity.

1.1 Related Work

In the *planar diameter improvement problem* (PDI), the input consists of a planar graph G and a positive integer d and the goal is to determine whether one can add edges to G in such a way that the resulting graph is planar and has diameter at most d. Note that the difference between EPDI and PDI is that in the former we are given an embedding that must be preserved, while in the latter no embedding is given at the input. Recently, using automata theoretic techniques, the second author was able to provide strongly uniform FPT and XP algorithms to many improvement problems where the parameter to be improved is definable in counting monadic second order logic [6]. In particular, when specialized to the PDI problem, the techniques in [6] yield a strongly uniform algorithm that solves PDI in time $f(d) \cdot 2^{O(\Delta \cdot d)} \cdot |V(G)|^{O(d)}$, where f is a computable function and Δ is the maximum degree of the input graph G. Nevertheless, the problem of determining whether PDI admits a strongly uniform algorithm running in time $f_1(d) \cdot |V(G)|^{f_2(d)}$ for computable functions f_1 and f_2 is still open if no bound is imposed on the degree of the input graph. We note that currently it is not known either whether PDI is reducible to EPDI or whether EPDI is reducible to PDI in XP time. Therefore it is not clear if our algorithm for EPDI can be used to provide a strongly uniform XP algorithm for PDI. It is worth noting that no hardness results for either PDI or EPDI are known. Indeed, determining whether either of these problems is NP-hard is also a long-standing open problem.

While the techniques employed in [6] to tackle the PDI problem on graphs of bounded degree were automata theoretic, the techniques employed in the present work to tackle the EPDI problem on general graphs is based on dynamic programming. In particular, our main algorithm carefully exploits the view of separators in plane graphs as *nooses* - simple closed curves in the plane that touch the graph only in the vertices (see e.g. [2]). The terminology *noose* for such curves comes from the graph minors project of Robertson and Seymour [13]. Our algorithm processes nooses in a way reminiscent of the dynamic programming algorithm of Bouchitte and Todinca over potential maximal cliques [3]. Although this method has found numerous applications in the field of graph algorithms [7, 12, 11], this work is the first which apply these techniques in the context of completion problems.

For each fixed d, let $\mathcal{G}_d$ be the *subgraph-closure*[1] of the class of planar graphs of diameter at most d. Then clearly a graph G is a yes instance of PDI if and only if $G \in \mathcal{G}_d$. When considering the task of constructing strongly uniform algorithms for PDI, two general approaches come to mind. The first follows by observing that graphs in $\mathcal{G}_d$ have treewidth at most $O(d)$, and that for each fixed $d \in \mathbb{N}$, $\mathcal{G}_d$ is MSO definable. Therefore, one could try to devise an algorithm $\mathfrak{A}$ that takes an integer d as input and constructs an MSO formula φ_d defining $\mathcal{G}_d$. With such a formula φ_d in hands, one could apply Courcelle's model checking theorem to determine whether a given graph G is an yes instance of PDI. The existence of such an algorithm $\mathfrak{A}$ is however an open problem. We note that one can easily define by induction on d an MSO sentence ϕ_d which is true on a graph G if and only if G is planar and has diameter at most d. Nevertheless, it is not clear how to use ϕ_d to construct φ_d. It is worth noting that there is no algorithm that takes an MSO sentence φ as input and constructs a sentence φ' defining the subgraph closure of the models of φ. For instance, let $\mathcal{L} = \{L_n\}_{n\in\mathbb{N}}$ be the family of ladder graphs, where L_n is the ladder with n steps. It is easy to see that $\mathcal{L}$ is MSO definable and every graph in $\mathcal{L}$ has treewidth at most 2. Nevertheless, the subgraph closure of $\mathcal{L}$ does not have finite index, and therefore such closure is *not* MSO definable.

The second approach is based on the fact that for each fixed $d \in \mathbb{N}$, the set of graphs $\mathcal{G}_d$ is minor closed. In particular, this implies that the class $\mathcal{G}_d$ can be characterized by a finite set $\mathcal{M}_d$ of forbidden minors. Therefore, one could try to devise an effective algorithm that takes an integer $d \in \mathbb{N}$ as input and gives as output the list of all forbidden minors in $\mathcal{M}_d$. By using the fact that minor-freeness can be tested FPT time, such an algorithm $\mathfrak{A}$ would solve PDI in FPT time. We observe however that the problem of listing the elements of $\mathcal{M}_d$ may be much more difficult than the problem of solving PDI in FPT time. It is worth noting that Adler, Kreutzer and Grohe have shown that if a minor-free graph property $\mathcal{P}$ is MSO definable and has constant treewidth, then one can effectively enumerate the set of forbidden minors for $\mathcal{P}$ [1]. Nevertheless, the problem with this approach is that it is not clear how to construct an MSO sentence φ_d defining $\mathcal{G}_d$.

2 Preliminaries

Graphs: For each $n \in \mathbb{N}$ we let $[n] = \{1, ..., n\}$. For each finite set S we let $\mathcal{P}(S, 2) = \{\{u, v\} \subseteq S \mid u \neq v\}$ be the set of unordered pairs of elements from S. In this work, a graph is a pair $G = (V(G), E(G))$ where $V(G)$ is a finite set of vertices and $E(G) \subseteq \mathcal{P}([n], 2)$ is a set of undirected edges. A path of length m in G is a sequence $p = v_0v_1...v_m$ of distinct vertices where for each $i \in \{0, ..., m-1\}$, $\{v_i, v_{i+1}\} \in E(G)$. We say that v_0 and v_m are the endpoints of p. The distance $dist(u, v)$ between vertices u and v is defined as the length of the shortest path with endpoints u and v. The diameter of G, is defined as $d(G) = \max_{u,v} dist(u, v)$.

Embeddings: A *simple arc* in $\mathbb{R}^2$ is a subset $\alpha \subseteq \mathbb{R}^2$ that is a homeomorphic image of the closed real interval $[0, 1]$. We let $endpts(\alpha)$ be the endpoints of α, and $Int(\alpha) = \alpha \backslash endpts(\alpha)$ be the interior of α. We let $\mathcal{A}$ be the set of simple arcs in $\mathbb{R}^2$. A planar embedding of G is a map $g : V(G) \cup E(G) \to \mathbb{R}^2 \cup \mathcal{A}$ that assigns a point $g(v) \in \mathbb{R}^2$ with each vertex $v \in V(G)$ and

[1] Note that the class of planar graphs of diameter at most d is closed under contractions but *not* under subgraphs, since removing edges may increase the diameter.

a simple arc $g(\{u,v\}) \in \mathcal{A}$ with each edge $\{u,v\}$ in such a way that the following conditions are satisfied.

1. For each $u, v \in V(G)$, $g(u) \neq g(v)$.
2. For each $\{u,v\} \in E(G)$, $\{g(u), g(v)\}$ are the endpoints of the simple arc $g(\{u,v\})$.
3. For each $\{u,v\} \in E(G)$, and each $w \in V(G)$ such that $w \neq u$ and $w \neq v$, $g(w) \notin g(\{u,v\})$.
4. For each $\{u,v\}, \{u',v'\} \in E(G)$, $Int(g(\{u,v\})) \cap Int(g(\{u',v'\})) = \emptyset$.

Intuitively, a planar embedding of a graph G is a drawing of G on the plane where each vertex v is represented by a point and each edge e is represented by a non self-intersecting curve that connects the points corresponding to the endpoints of e, and no crossings are allowed. A *plane graph* is a pair $G^g = (G, g)$ where G is a graph and g is a planar embedding of G. For technical reasons, in this work we assume that the origin $(0,0) \in \mathbb{R}^2$ is distinct from $g(v)$ for each $v \in V(G)$ and does not belong to $g(e)$ for each edge $e \in E(G)$.

Plane Completion: Let G and H be graphs. We say that G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. If G^g and H^h are plane graphs, then we say that G^g is a *plane subgraph* of H^h if G is a subgraph of H, $g|_{V(G)} = h|_{V(G)}$ and $g_{E(G)} = h|_{E(G)}$. Alternatively, we say that H^h is a *plane completion* of G^g (Figure 3). We say that such a completion H^h is triangulated if each face of H^h has three vertices.

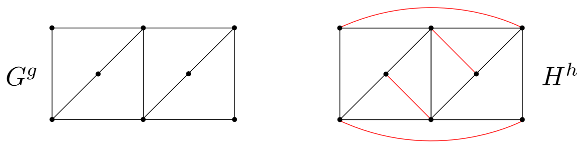

Figure 1 Left: a plane graph G of diameter 3. Right: a plane completion of G of diameter 2.

Combinatorial face: Let G^g be a plane graph. We say that a point $p \in \mathbb{R}^2$ is *independent* of G^g if $p \neq g(v)$ for every $v \in V(G)$ and $p \notin g(e)$ for every $e \in E(G)$. We say that an independent point p *reaches* a point $p' \in \mathbb{R}^2$ if there is a simple arc α with endpoints p and p' that does not contain any vertex in $g(V) \backslash \{p, p'\}$ and does not intersect the interior of any arc in $g(E)$. If p is an independent point then we let $\mathcal{F}(G^g, p)$ be the subgraph of G whose vertex set is

$$V(\mathcal{F}(G^g, p)) = \{v \in V(G) \mid p \text{ reaches } g(v)\}$$

and whose edge set is

$$E(\mathcal{F}(G^g, p)) = \{e \in E(G) \mid p \text{ reaches each point in } g(e)\}.$$

We let $b(p)$ be a boolean value that is 0 if the origin $(0,0)$ is reachable from p, and 1 otherwise. We say that a pair $F = (X, b)$, where X is a subgraph of G and $b \in \{0,1\}$, is a *combinatorial face* if there exists a $p \in \mathbb{R}^2$ such that $X = \mathcal{F}(G^g, p)$ and $b = b(p)$. We note that by $X = \mathcal{F}(G^g, p)$ we mean $V(X) = V(\mathcal{F}(G^g, p))$ and $E(X) = E(\mathcal{F}(G^g, p))$. For instance, if G^g is a plane graph where G is a tree, then the unique combinatorial face of G^g is the pair $F = (G, 1)$. On the other hand, if G is a cycle, then G^g has two faces: $F_1 = (G, 0)$ and $F_2 = (G, 1)$. We write $F(G^g)$ to denote the set of all faces of G^g. We note that $F(G^g)$ has at most $O(|V(G)|)$ faces.

We say that two embedded versions G^g and $G^{g'}$ of a graph G are equivalent if $F(G^g) = F(G^{g'})$. We write $G^g \equiv G^{g'}$ to denote that G^g and $G^{g'}$ are equivalent.

3 Nooses

▶ **Definition 1.** Let G^g be a plane graph. A G^g*-noose* is a subset $\eta \subseteq \mathbb{R}^2$ homeomorphic to the unit circle S_1 such that $\eta \cap Int(g(uv)) = \emptyset$ for every edge $uv \in E(G)$.

We note that if η is a G^g-noose, the intersection $\eta \cap g(V(G))$ may be non-empty. We let $V(\eta) = \{v \in V(G) \mid g(v) \in \eta\}$ be the set of vertices of G whose image lies in the noose η. The size of η is defined as $|\eta| = |V(\eta)|$. We let $\hat{V}(\eta)$ be the set of vertices lying in the closed interior of η.

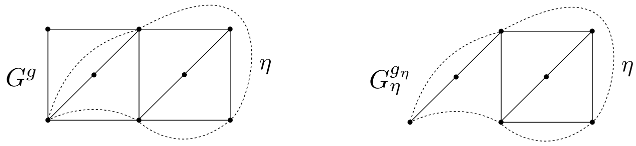

Figure 2 Left: a plane graph G and one of its nooses η. Right: the graph $G_\eta^{g_\eta}$ that lies in the interior of η.

Combinatorial cycle: Let Σ be a finite set. We let Σ^k be the set of sequences of length k over Σ. If $a_0a_1...a_{k-1}$ is a sequence in Σ^k, then we let

$$[a_0a_1...a_{k-1}] = \{a_{j_0}a_{j_1}...a_{j_{k-1}} \mid \exists l \in \{0,...,k-1\}\forall i \in \{0,...,k-1\},\ j_i = i + l \ mod\ k\}.$$

be the set of all cyclic shifts of the string $a_0a_1...a_{k-1}$. We say that the set $[a_0a_1...a_{k-1}]$ is a *combinatorial cycle* over Σ.

Noose Type: Let G^g be a plane graph and $\Sigma(G) = V(G) \cup F(G)$. A G^g*-noose type* is a cycle $\tau = [v_0F_0v_1F_1...v_{r-1}F_{r-1}]$ over $\Sigma(G)$ where $\{v_0,...,v_{r-1}\} \subseteq V(G)$, $F_0,...,F_{r-1} \subseteq F(G^g)$, and $\{g(v_i), g(v_{i+1 \bmod r})\} \subseteq F_i$ for each $i \in \{0,1,...,r-1\}$. We say that a G^g noose η is compatible with τ if there exist simple arcs $\ell_0,...,\ell_{r-1}$ satisfying the following properties.

1. $\eta = \bigcup_{i\in\{0,...,r-1\}} \ell_i$.
2. For each $i \in \{0,...,r-1\}$, $endpts(\ell_i) = \{g(v_i), g(v_{i+1 \bmod r})\}$.
3. For each $i \in \{0,...,r-1\}$, $\ell_i \cap \ell_{i+1 \bmod r} = v_{i+1 \bmod r}$.

We say that two G^g-nooses η_1 and η_2 are equivalent if there exists a G^g-noose type τ such that both η_1 and η_2 are compatible with τ. We note that uncountably many G^g-nooses may be compatible with a given G^g-noose type τ.

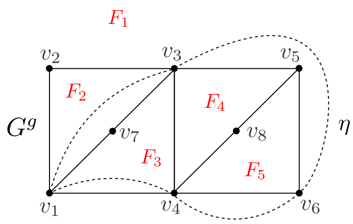

Figure 3 The type of the noose type η is the cycle $[v_1F_2v_3F_1v_6F_1v_4F_3]$. Note that the segment of η between v_1 and v_3 lies in the area delimited by the face F_2, the segment between v_3 and v_6 lies in the area delimited by face F_1 and so on.

Let γ be a subset of $\mathbb{R}^2$ homeomorphic to the unit circle S_1. We say that a point $p \in \mathbb{R}^2$ belongs to the closed interior of γ if $\alpha \cap \gamma \neq \emptyset$ for every simple arc α with $endpts(\alpha) = \{(0,0), p\}$. We let $\hat{\gamma}$ be the set of points in the closed interior of γ. If $G^g = (G, g)$ is a plane graph and η is a G^g-noose, then we let $G_\eta^{g_\eta} = (G_\eta, g_\eta)$ be the plane graph where

1. $V(G_\eta) = \{v \in V(G) \mid g(v) \in \hat{\eta}\}$,
2. $E(G_\eta) = \{uv \in E(G) \mid g(uv) \subset \hat{\eta}\}$,
3. $g_\eta : V(G_\eta) \cup E(G_\eta) \to \mathbb{R}_2 \cup \mathcal{A}$ with $g_\eta|_{V(G_\eta)} = g|_{V(G_\eta)}$ and $g_\eta|_{E(G_\eta)} = g|_{E(G_\eta)}$.

Intuitively, the graph $G^{g_\eta}_\eta$ is the plane subgraph of G^g that lies in the closed interior of η.

▶ **Observation 1.** Let G^g be a plane graph and η_1 and η_2 be equivalent G^g-nooses. Then $G^{g_{\eta_1}}_{\eta_1} = G^{g_{\eta_2}}_{\eta_2}$.

▶ **Definition 2.** Let G^g be a plane graph and τ be a G^g-noose type. We say that a plane completion H^h of G^g_τ is τ-respecting if there is a G^g-noose η of type τ which is also a H^h-noose.

We note that in Definition 2, although the G^g-noose type of η is τ, the H^h-noose type of η is not necessarily τ since H^h may have more faces than G^g.

4 Representative Sets

Let G^g be a plane graph and η be a G^g-noose. We say that a plane completion H^h of G^g is *η-respecting* if η is also an H^h-noose. We say that a plane graph X^x is a (G^g, η)-completion if the following conditions are satisfied.

1. η is a X^x-noose.
2. $X^x = X^{x_\eta}_\eta$.
3. X^x is a plane completion of $G^{g_\eta}_\eta$.

Intuitively, (G^h, η)-completion is a plane completion X^x of $G^{h_\eta}_\eta$ where all vertices and edges are drawn inside η.

▶ **Proposition 1.** If H^h is an η-respecting plane completion of G^g, then $H^{g_\eta}_\eta$ is a (G^g, η)-completion.

If G^g is a plane graph and H^h is a plane subgraph of G^g then we let $G^g - H^h$ be the plane graph Y^y where $V(Y) = V(G)$, $E(Y) = E(G) \backslash E(H)$ and $y = g|_{V(Y)}$. In other words, $G^g - H^h$ is the graph obtained by deleting from G the edges which are shared with H. On the other hand, let G^g and H^h be plane graphs such that $V(H) \subseteq V(G)$, $Int(h(e)) \cap g(e') = \emptyset$ for every $e \in E(H) \backslash E(G)$ and every $e' \in E(G)$, and such that $h(e) = g(e)$ for every $e \in E(G) \cap E(H)$. then we let $G^g + H^h$ be the plane graph Y^y with vertex set $V(Y) = V(G)$, edge set $E(Y) = E(G) \cup E(H)$, and embedding y such that $y|_{V(G)} = g|_{V(G)}$, $y|_{E(G)} = g|_{E(G)}$ and $y|_{E(H)\backslash E(G)} = h|_{E(H)\backslash E(G)}$. In other words, $G^g + H^h$ is the graph obtained by adding all edges in $E(H)\backslash E(G)$ to G and by drawing these edges in the plane according to h.

We say that H^h is a η-respecting diameter-d plane completion of G^g if H^h is a η-respecting plane completion of G^g of diameter at most d.

▶ **Definition 3.** Let G^g be a plane graph and η be a G^g-noose. Let A and B be sets of (G^g, η)-completions. We say that A represents B if for every diameter-d η-respecting plane completion H^h of G^g, such that $H^{h_\eta}_\eta \in B$, there exists some $X^x \in A$ such that $(H^h - H^{h_\eta}_\eta) + X^x$ is also a diameter-d plane completion of G^g.

Let G^g be a plane graph and η be a G^g-noose. We let $V(\eta) = \{v \in V(G) \mid g(v) \in \eta\}$ be the set of vertices of G whose image under g lies on η, and $\hat{V}(\eta)$ be the set of vertices of G that lie in the closed interior of η.

Let X^x be a (G^g, η)-completion. The truncated distance between any two vertices v, v' of $\hat{V}(\eta)$, denoted by $d(X^x, v, v')$ is defined as the distance between v and v' in X^x if this distance is at most d, and ∞ otherwise. We let $D(X^x) = [d(X^x, v, v')]_{v,v' \in V(\eta)}$, be the matrix of truncated distances between any two vertices in $V(\eta)$. For any vertex v in $\hat{V}(\eta)$, we let $D(X^x, v) = [d(X^x, v, v')]_{v' \in V(\eta)}$ be the vector of distances between v and the vertices whose image lie in the noose η. We say that two vertices u and u' in $\hat{V}(\eta)$ are unresolved if their distance in X^x is greater than d. For each pair of unresolved vertices we let $D(X^x, u, u') = (D(X^x, u), D(X^x, u'))$ be the pair of distance vectors from u to $V(\eta)$ and from u' to $V(\eta)$ respectively. We let

$$\mathcal{D}(X^x) = \{D(X^x, u, u') \mid (u, u') \text{ is an unresolved pair}\}.$$

The signature of X^x is defined as follows.

$$\mathcal{S}(X^x) = (D(X^x), \mathcal{D}(X^x)). \quad (1)$$

▶ **Proposition** 2. Let $|V(\eta)| \leq 8d$. Then there exist at most $2^{d^{O(d)}}$ distinct signatures.

▶ **Lemma 4.** *Let H^h be a plane completion of G^g of diameter at most d and let X^x be a (G^g, η)-completion such that $\mathcal{S}(H_\eta^{h_\eta}, \eta) = \mathcal{S}(X^x, \eta)$. Then if H^g has diameter at most d, $(H^h - H_\eta^{h_\eta}) + X^x$ has diameter at most d.*

▶ **Lemma 5.** *Let G^g be a plane graph, η be a G^g-noose and B be a set of (G^g, τ)-completions. Then one can construct in time $2^{d^{O(d)}} \cdot |B|$ a set $Trunc(B)$ of (G^g, h)-completions such that $|Trunc(B)| \leq d^{O(d)}$ and $Trunc(B)$ represents B.*

Proof. Let B be a set of (G^g, η) completions. For each signature S, let $B(S)$ be the set of all graphs H^h in B with $\mathcal{S}(H^h, \eta) = S$. Finally, let $Trunc(B)$ be the set obtained by selecting a unique graph H_S^h from set $B(S)$ whenever $B(S)$ is non-empty. Then by Lemma 4, $Trunc(B)$ represents B. ◀

5 Algorithm for Embedded Planar Diameter Improvement

In this section we will devise an algorithm that takes a plane graph G^g and a positive integer d as input and determines in time $|V(G)|^{O(d)}$ whether G^g has a diameter-d plane completion H^h. In the reminder of this section we assume that the input plane graph G^g and the input positive integer d are fixed.

▶ **Definition 6.** Let η be a G^g-noose . We define the following set.

$$\mathcal{F}_\eta = \{H_\eta^{h_\eta} \mid H^h \text{ is a diameter-}d \text{ plane completion of } G^g\}.$$

Intuitively, $\mathcal{F}_\eta$ is the set of all plane completions of the graph $G_\eta^{g_\eta}$ that can be extended to some diameter-d plane completion of the graph G^g. For each G^g-noose η we will define a family $\tilde{\mathcal{F}}_\eta \subseteq \mathcal{F}_\eta$ of (G^g, η)-completions such that $\tilde{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$ and $|\tilde{\mathcal{F}}_\eta| \leq 2^{d^{O(d)}}$.

We will define a partial order on nooses as follows. First, for each noose η, we consider the following triple.

$$\phi(\eta) = [|V(G_\eta^{g_\eta})|, |E(G_\eta^{g_\eta})|, -|\eta|].$$

We set $\eta < \eta'$ if and only if $\phi(\eta) < \phi(\eta')$. In other words, a noose η is smaller than a noose η' if the closed interior of η has less vertices than the closed interior of η', or if these interiors have the same number of vertices, but the first has less edges, or if both interiors have the same number of edges and vertices and the first noose has more vertices than the latter nose. Note that there is an inversion in the third coordinate, since the larger the noose-size, the lesser is the order.

We will compute $\tilde{\mathcal{F}}_\eta$ under the assumption that $\tilde{\mathcal{F}}_{\eta'}$ has been computed for every η' such that $\phi(\eta') < \phi(\eta)$. There are three cases to be considered. In all cases the size of the involved nooses will be at most $8d$. In the first case, assuming that the size of η is at most $8d$, we will show how to compute $\tilde{\mathcal{F}}_\eta$ under the assumption that $\tilde{\mathcal{F}}_{\eta'}$ has been computed for every noose η' whose closed interior has fewer edges than the one of η. The second case concerns nooses of size strictly less than $8d$. In this case, we will show how to compute $\tilde{\mathcal{F}}_\eta$ assuming that we have computed $\tilde{\mathcal{F}}_{\eta'}$ for every noose η' of size $|\eta| + 1$ whose closed interior is identical to the one of η. Finally, the most important case is the third, in which nooses have size exactly $8d$. In this case we show how to compute $\tilde{\mathcal{F}}_\eta$ assuming we have computed $\tilde{\mathcal{F}}_{\eta'}$ for every noose of size $8d$ whose closed interior is strictly contained in the closed interior of η. We note that we only need to consider one noose η_τ for each noose-type τ. Therefore, the number of nooses to be considered will be upper bounded by $n^{O(d)}$.

5.1 Case One

Let G^g be a plane graph and η be a G^g-noose with $|\eta| < 8d$. We say that an edge $uv \in E(G)$ is *parallel* to η if there exists a simple arc $\ell \in \mathcal{A}$ such that the following conditions are satisfied.

1. $\ell \subseteq \eta$.
2. $endpts(\ell) = \{g(u), g(v)\}$.
3. Let $\alpha = \ell \cup g(uv)$ and let $\hat{\alpha}$ be the closed interior of α.
   - **a.** $\hat{\alpha} \cap g(V(G)) = \{g(u), g(v)\}$.
   - **b.** $\hat{\alpha} \cap g(u'v') \subseteq \{g(u), g(v)\}$ for each $u'v' \in E(G)$.

In other words, uv is parallel to η if there is a simple arc $\ell \subset \eta$ with endpoints $\{g(u), g(v)\}$ such that the whole region delimited by ℓ and η only intersects the drawing of G in the points $g(u)$ and $g(v)$ (Figure 4).

We say that an edge $uv \in E(G)$ is *internally* (resp. *externally*) parallel to η if uv is parallel to η and $g(uv) \subset \hat{\eta}$ (resp. $g(u,v) \cap \hat{\eta} = \{g(u), g(v)\}$). We note that if an edge uv is parallel to η, then uv is either internally or externally parallel to η. Let G^g be a plane graph and $uv \in E(G)$. We let $G^g - uv$ be the plane graph which is obtained by deleting the edge uv from $E(G)$ and by restricting the mapping g to the remaining edges.

▶ **Proposition** 3. Let η be a G^g-noose, and let uv be an edge in $E(G)$ such that uv is internally parallel to η. Then there exists a G^g-noose η' such that $V(\eta) = V(\eta')$, uv is externally parallel to η' and $G_{\eta'}^{g_{\eta'}} = G_{\eta}^{g_{\eta}} - uv$.

Proposition 3 is illustrated in Figure 4.

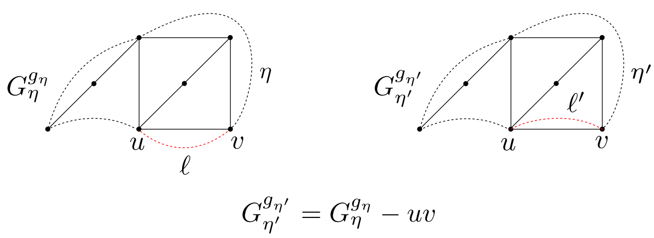

Figure 4 The edge uv is internally parallel to η and externally parallel to η'. Note that the graph $G_{\eta'}^{g_{\eta'}}$ is equal to the graph $G_{\eta}^{g_{\eta}}$ minus the edge uv. Intuitively, the noose η' is obtained from η by deleting the arc ℓ and by gluing the arc ℓ' in its place.

▶ **Lemma 7.** *Let uv be an edge in $E(G)$, and let η be a G^g-noose such that uv is internally parallel to η. Let η' be a G^g-noose such that $V(\eta) = V(\eta')$, $G_{\eta'}^{g_{\eta'}} = G_{\eta}^{g_{\eta}} - uv$ and uv is externally parallel to η'. Note that such a noose η' exists by Proposition 3. Suppose that $X^x \in \mathcal{F}_{\eta}$. Then there exists $Y^y \in \mathcal{F}_{\eta'}$ such that $Y^y = X^x - uv$.*

Let Y^y be a (G^g, η)-completion and uv be an edge in $E(G) \backslash E(Y)$ such that $u, v \in V(\eta)$. We let $Y^y + uv$ be the plane graph X^x where $V(X) = V(Y)$, $E(X) = E(Y)$, $x|_{V(G) \cup E(G)} = y$ and $x(uv) = g(uv)$.

▶ **Lemma 8.** *Let uv be an edge in $E(G)$, and let η and η' be G^g-nooses such that $V(\eta) = V(\eta')$ and $G_{\eta}^{g} = G_{\eta'}^{g} + uv$. Assume that $\tilde{\mathcal{F}}_{\eta'}$ represents $\mathcal{F}_{\eta'}$. Then $\tilde{\mathcal{F}}_{\eta} = \tilde{\mathcal{F}}_{\eta'} + uv$ represents $\mathcal{F}_{\eta}$.*

5.2 Case Two

Let $k < 8d$ and let η be a G^g-noose of type $\tau = [v_0 F_0 v_1 F_1 ... v_{k-1} F_{k-1}]$. A trivial extension of η is a noose G^g-noose η' of type

$$\tau = [v_0 F_0 v_1 F_1 ... v_j F_j u F_j v_{j+1} ... v_{k-1} F_{k-1}]$$

for some $j \in \{0, ..., k-1\}$ and some u which belongs to face F_j and to the closed interior of η. Note that if η' is a trivial extension of η, then η and η' have identical closed interiors. In Figure 5 we depict two trivial extensions η' and η'' of a noose η.

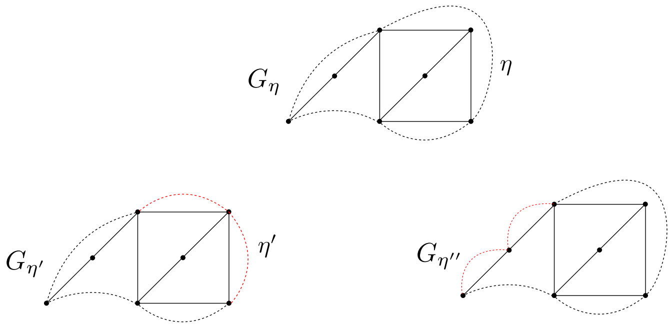

Figure 5 The nooses η' and η'' are trivial extensions of η. The red dashed region indicates where a new vertex was added. Note that $G^g_\eta = G^g_{\eta'} = G^g_{\eta''}$.

We say that a noose η is extensible if $|V(\eta)| < 8d$ and if η has at least one trivial extension. We let $\text{Ext}(\eta)$ be the set of (equivalence classes of) trivial extensions of η. The following equality is immediate.

$$\mathcal{F}_\eta = \bigcup_{\eta' \in \text{Ext}(\eta)} \mathcal{F}_{\eta'} \tag{2}$$

For each extensible noose, we consider the following set.

$$\hat{\mathcal{F}}_\eta = \bigcup_{\eta' \in \text{Ext}(\eta)} \tilde{\mathcal{F}}_{\eta'} \tag{3}$$

Now we have the following lemma.

▶ **Lemma 9.** *Let η be an extensible noose. Suppose that $\tilde{\mathcal{F}}_{\eta'}$ represents $\mathcal{F}_{\eta'}$ for every $\eta' \in \text{Ext}(\eta)$. Then $\hat{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$.*

Note that the number of extensions of a noose η may be linear in $|V(G)|$. Therefore, the number of plane graphs in $\hat{\mathcal{F}}_\eta$ may be linear in $|V(G)|$. To decrease the size of this family, we apply the *Trunc* operator introduced in Section 4.

▶ **Lemma 10.** *Let η be an extensible noose and let $\tilde{\mathcal{F}}_\eta = Trunc(\hat{\mathcal{F}}_\eta)$. Then $\tilde{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$.*

5.3 Case Three

In this section we will deal with the case in which $|\eta| = 8d$. Let G^g be a plane graph. A face-vertex sequence is a sequence $X = F_1 v_1 F_2 ... v_{r-1} F_r$ where $r \geq 1$, $\{v_1, ..., v_{r-1}\} \subseteq V(G)$, $\{F_1, ..., F_r\} \subseteq F(G)$, and for each $i \in \{1, ..., r-2\}$, v_i and v_{i+1} are in F_i. We note that any noose type $\tau = [v_0 F_0 v_1 ... v_m F_m]$ where $m \geq 1$ can be written as $\tau = [v_0 X v_r Y]$ where $X = F_1 v_1 F_2 ... v_{r-1} F_{r-1}$ and $Y = F_r v_{r+1} F_{r+1} ... v_m F_m$ are face-vertex sequences.

The *reverse* of a face-vertex sequence $X = F_1 v_1 F_2 ... v_{r-1} F_r$ is the face-vertex sequence $X^R = F_r v_{r-1} ... F_2 v_1 F_1$. Let τ_1 and τ_2 be noose types. We say that τ_1 and τ_2 are *summable* if there is a unique maximal[2] face-vertex sequence $X = X(\tau_1, \tau_2)$ with the property that there exist

[2] Maximal with respect to length.

vertices v, v' and face-vertex sequences Y and Z such that $\tau_1 = [vYv'X]$ and $\tau_2 = [v'ZvX^R]$. If this is the case, the sum of τ_1 with τ_2 is defined as $\tau_1 \oplus \tau_2 = [vYv'Z]$. We let $V(X(\tau_1, \tau_2))$ denote the vertices which lie in the face vertex sequence $X(\tau_1, \tau_2)$.

We note that a noose η has type $\tau_1 \oplus \tau_2$ if and only if there exist vertices v, v' and simple arcs $\ell_1, \ell_1', \ell_2, \ell_2'$ with endpoints $\{v, v'\}$ satisfying the following properties.

1. $\ell_1 \cap \ell_1' = \ell_2 \cap \ell_2' = \ell_1 \cap \ell_2 = \{v, v'\}$.
2. $g(V(X(\tau_1, \tau_2))) \subseteq \ell_1' \cap \ell_2'$.
3. $\eta = \ell_1 \cup \ell_2$.
4. $\eta_1 = \ell_1 \cup \ell_1'$ is a noose of type τ_1.
5. $\eta_2 = \ell_2 \cup \ell_2'$ is a noose of type τ_2.

Intuitively, η is obtained from η_1 and η_2 by the following process. First, we delete the interior of ℓ_1' from η_1 to obtain the segment ℓ_1, and we delete the interior of ℓ_2' from η_2 to obtain the segment ℓ_2. Subsequently, we glue ℓ_1 with ℓ_2 along their common endpoints in order to obtain the noose η. We note that if $\eta = \eta_1 \oplus \eta_2$ then $G_\eta^{g_\eta} = G_{\eta_1}^{g_{\eta_1}} \cup G_{\eta_2}^{g_{\eta_1}}$.

Let η be a G^g-noose with $|\eta| = 8d$ and let H^h be a triangulated η-respecting diameter-d plane completion of G^g. Let $V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$ be a partition of $V(\eta)$ where for each $i \in \{1, 2, 3, 4\}$, V_i has d consecutive vertices. Let $\hat{V}_1 \dot\cup \hat{V}_2 \dot\cup \hat{V}_3 \dot\cup \hat{V}_4$ be a partition of the vertex set of $H_\eta^{h_\eta}$ where for each $v \in V(H_\eta^{h_\eta})$,

$$v \in \hat{V}_i \;\Leftrightarrow\; \left[dist(v, V_i) < \min_{j<i} dist(v, V_j) \;\wedge\; dist(v, V_i) \le \min_{j>i} dist(v, V_j) \right].$$

Intuitively, the set of vertices incident with the noose η is partitioned into four consecutive sections of size d: north (V_1), east (V_2), south (V_3) and west (V_4). Subsequently, each vertex $v \in V(G_\eta^g)$ is classified as being north ($\hat{V}_1$), east ($\hat{V}_2$), south ($\hat{V}_3$) or west ($\hat{V}_4$) according to whether v is closer to a northern, eastern, southern or western vertex from the noose. In the case in which a vertex v is as close to V_i as it is to V_j, for some $i \neq j$, ties are broken by considering that north is smaller than east, which is smaller than south, which is smaller than west. We note that the way in which we have decided to break ties is completely arbitrary.

▶ **Lemma 11.** *There is an edge $uv \in E(H_\eta^{h_\eta})$ such that either $u \in \hat{V}_1$ and $v \in \hat{V}_3$, or $u \in \hat{V}_2$ and $v \in \hat{V}_4$.*

▶ **Lemma 12.** *At least one of the following statements must be satisfied.*

1. *There is a path of length at most $2d + 1$ between a vertex in V_1 to a vertex in V_3.*
2. *There is a path of length at most $2d + 1$ between a vertex in v_2 and a vertex in v_4.*

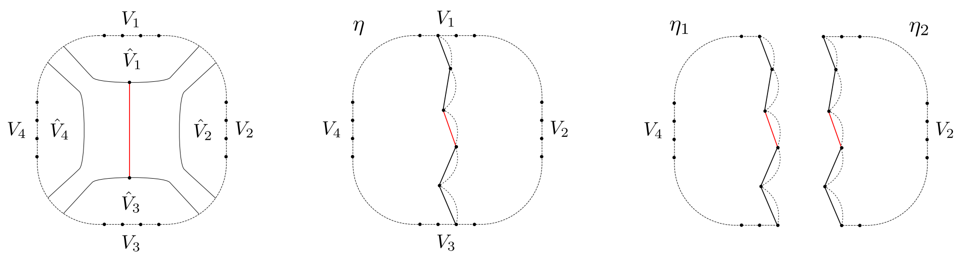

Figure 6 Left: A noose η with $8d$ vertices, and the sets V_i and $\hat{V}_i$. Either there is an edge from a maximal element of $\hat{V}_1$ to a maximal element of $\hat{V}_3$, or an edge between a maximal element of $\hat{V}_2$ and a maximal element of $\hat{V}_4$. Middle: A path from V_1 to V_3 is depicted. Right: the path from V_1 to V_3 can be used to show that $\eta = \eta_1 \oplus \eta_2$ for nooses η_1 and η_2 where can be split into nooses $\eta_1 \oplus \eta_2$.

▶ **Lemma 13.** *Let H^h be a η-respecting diameter-d plane triangulated completion of G^g. Then there exist G^g-nooses η_1 and η_2 satisfying the following properties.*
1. $|V(\eta_1)| \leq 8d$ *and* $|V(\eta_2)| \leq 8d$.
2. $\eta = \eta_1 \oplus \eta_2$.
3. H^h *is both* η_1*-respecting and* η_2*-respecting.*
4. $H_{\eta}^{h_\eta} = H_{\eta_1}^{h_{\eta_1}} \cup H_{\eta_2}^{h_{\eta_2}}$.

Let η, η_1 and η_2 be G^g nooses such that $\eta = \eta_1 \oplus \eta_2$. Then we define the following set.

$$\mathcal{F}_{\eta_1} \oplus \mathcal{F}_{\eta_2} = \{X^x \cup Y^y \mid X^x \in \mathcal{F}_{\eta_1},\ Y^y \in \mathcal{F}_{\eta_2}\}.$$

Now, we have that Lemma 13 implies the following equality whenever $|\eta| = 8d$.

$$\mathcal{F}_\eta = \bigcup_{\eta=\eta_1\oplus\eta_2} \mathcal{F}_{\eta_1} \oplus \mathcal{F}_{\eta_2} \tag{4}$$

Now we have the following lemma.

▶ **Lemma 14.** *Let η be a G^g-noose with $|\eta| = 8d$, and assume that for every two G^g-nooses η_1 and η_2 such that $\eta = \eta_1 \oplus \eta_2$ we have that $\tilde{\mathcal{F}}_{\eta_1}$ represents $\mathcal{F}_{\eta_1}$ and $\tilde{\mathcal{F}}_{\eta_2}$ represents $\mathcal{F}_{\eta_2}$. Then $\mathcal{F}_\eta$ is represented by the following set.*

$$\tilde{\mathcal{F}}_\eta = \mathit{Trunc}\left(\bigcup_{\eta=\eta_1\oplus\eta_2} \tilde{F}_{\eta_1} \oplus \tilde{F}_{\eta_2}\right) = \mathit{Trunc}\left(\bigcup_{\eta=\eta_1\oplus\eta_2} \{X^x \cup Y^y \mid X^x \in \tilde{\mathcal{F}}_{\eta_1},\ Y^y \in \tilde{\mathcal{F}}_{\eta_2}\}\right) \tag{5}$$

Proof. Let H^h be a diameter-d plane completion of G^g, and let $H_\eta^{h_\eta} \in \mathcal{F}_\eta$. By Lemma 13, we have that there exist nooses η_1 and η_2 of length at most $8d$ such that $H_{\eta_1}^{h_{\eta_1}} \in \mathcal{F}_{\eta_1}$, $H_{\eta_2}^{h_{\eta_2}} \in \mathcal{F}_{\eta_2}$ and $H_\eta^{h_\eta} = H_{\eta_1}^{h_{\eta_1}} \cup H_{\eta_2}^{h_{\eta_2}}$. But by assumption we know that $\tilde{\mathcal{F}}_{\eta_1}$ represents $\mathcal{F}_{\eta_1}$ and that $\tilde{\mathcal{F}}_{\eta_2}$ represents $\mathcal{F}_{\eta_2}$. Therefore, there exists $X^x \in \tilde{\mathcal{F}}_{\eta_1}$ and $Y^y \in \tilde{\mathcal{F}}_{\eta_2}$ such that $(H^h - (H_{\eta_1}^{h_{\eta_1}} \cup H_{\eta_2}^{h_{\eta_2}})) + (X^x \cup Y^y)$ is also a diameter-d plane completion of G^g. This shows that the set

$$\hat{\mathcal{F}}_\eta = \bigcup_{\eta=\eta_1\oplus\eta_2} \tilde{F}_{\eta_1} \oplus \tilde{F}_{\eta_2} = \bigcup_{\eta=\eta_1\oplus\eta_2} \{X^x \cup Y^y \mid X^x \in \tilde{\mathcal{F}}_{\eta_1},\ Y^y \in \tilde{\mathcal{F}}_{\eta_2}\}$$

represents $\mathcal{F}_\eta$. Now by applying the operator *Trunc* from Lemma 5 we have that $\tilde{\mathcal{F}} = \mathit{Trunc}(\hat{\mathcal{F}})$ also represents $\mathcal{F}_\eta$. ◀

5.4 Algorithm

Now we summarize our algorithm for determining whether a given plane graph G^g admits a plane completion of diameter at most d. We assume that the graph has at least d vertices, since otherwise the graph has trivially a completion of diameter at most d. As a first step we enumerate all combinatorial faces of G^g, constructing in this way the set $F(G^g)$. We note that there exists at most $O(|V(G)|)$ such faces, and the set $F(G^g)$ can clearly be constructed in time $|V(G)|^{O(1)}$. As a second step, we enumerate all noose types containing at most $8d$ vertices. In other words, we enumerate all cycles $\tau = [v_0F_1v_1...v_{r-1}F_r]$ where $r-1 \leq 8d$, and verify if τ is a valid noose type. Since there are at most $O(|V(G)|)$ combinatorial faces, there are at most $|V(G)|^{O(d)}$ possible noose types with at most $8d$ vertices. Let $\mathcal{T}$ be the set of all such noose types. Now, for each noose type $\tau \in \mathcal{T}$, we select an arbitrary noose η_τ of type τ. We say that η_τ is a representative for τ. Let $\mathcal{N} = \{\eta_\tau\}_{\tau\in\mathcal{T}}$ be the set of all such nooses.

For each noose $\eta \in \mathcal{N}$, we will compute a set $\tilde{\mathcal{F}}_\eta$ containing at most $2^{d^{O(d)}}$ triangulated plane completions of the graph $G_\eta^{g_\eta}$. In particular the set $\tilde{\mathcal{F}}_\eta$ represents the set $\mathcal{F}_\eta$. The sets $\tilde{\mathcal{F}}_\eta$ are computed by dynamic programming.

In the base case, let $\mathcal{N}^0 \subseteq \mathcal{N}$ be the minimal elements of $\mathcal{N}$ with respect to the noose ordering defined in the beginning of Section 5. For each such minimal noose η^0, we have that $|V(\eta^0)| = 3$, and the graph $G_\eta^{g_\eta}$ has no edges. Therefore the set $\tilde{\mathcal{F}}_\eta$ can be constructed in constant time in this case.

Now assume that we are dealing with a non-minimal noose $\eta \in \mathcal{N}$ and that $\tilde{\mathcal{F}}_{\eta'}$ has been constructed for every $\eta' < \eta$. We will show how to construct $\tilde{\mathcal{F}}_\eta$. There are three cases to be considered.

1. If there is some edge $uv \in E(G)$ which is externally parallel to η, then we consider a noose η' such that $G_{\eta'}^{g_{\eta'}} = G_\eta^{g_\eta} - uv$. Then we set $\tilde{\mathcal{F}}_\eta = \tilde{\mathcal{F}}_{\eta'} + uv$.
2. If η is trivially extensible, then we let $\tilde{\mathcal{F}}_\eta = \mathit{Trunc}\left(\bigcup_{\eta' \in \mathrm{Ext}(\eta)} \tilde{\mathcal{F}}(\eta')\right)$.
3. If $|\eta| = 8d$, then we let $\tilde{\mathcal{F}}_\eta = \mathit{Trunc}\left(\bigcup_{\eta=\eta_1\oplus\eta_2} \tilde{\mathcal{F}}_{\eta_1} \oplus \tilde{\mathcal{F}}_{\eta_2}\right)$

Note that in any of the three cases above the time necessary to construct the set $\tilde{\mathcal{F}}_\eta$ from previously computed $\tilde{\mathcal{F}}_{\eta'}$ is at most $2^{d^{O(d)}} \cdot |V(G)|^{O(d)}$, since there are at most $|V(G)|$ nooses in $\mathcal{N}$ and each $\tilde{\mathcal{F}}_{\eta'}$ has at most $2^{d^{O(d)}}$ elements. This implies that the computation of $\tilde{\mathcal{F}}_\eta$ for every $\eta \in \mathcal{N}$ also takes time at most $2^{d^{O(d)}} \cdot |V(G)|^{O(d)}$.

Now let η be a maximal noose in $\mathcal{N}$. Then we have that $|\eta| = 3$ and that the graph $G_\eta^{g_\eta} = G^g$. Therefore, we have that G^g admits a diameter-d plane completion if and only if the set $\tilde{\mathcal{F}}_\eta$ is non-empty. Additionally, if this is the case, then any graph in $\tilde{\mathcal{F}}_\eta$ is a diameter-d plane completion of G^g.

5.5 k-Outerplanar Plane Diameter Improvement

The next theorem states that for each fixed k, the embedded k-outerplanar diameter improvement problem is strongly uniformly fixed parameter tractable with respect to the parameter d.

▶ **Theorem 15.** *There is an algorithm $\mathfrak{A}$ that takes as input, positive integers d, k, and a plane graph G^g, and determines in time $2^{d^{O(d)}} \cdot |V(G)|^{O(k)}$ whether G has a k-outerplanar plane completion H^h of diameter at most d.*

Proof. The algorithm for solving the embedded k-outerplanar diameter improvement problem is almost identical to the one to solve the embedded planar diameter improvement problem. The only difference is that we need to restriction our attention to nooses of size at most $8k$, instead of $8d$ as in the previous algorithm. Therefore, the algorithm runs in time $2^{d^{O(d)}} \cdot |V(G)|^{O(k)}$ instead of in time $2^{d^{O(d)}} \cdot |V(G)|^{O(d)}$.

The reason we need to consider nooses of size at most $8k$ is due to the fact that since the searched completion must be k-outerplanar, and can be assumed to be triangulated, the distance between any two vertices lying on a noose η is at most $2k$. Therefore, the proof of Lemma 12 may be adapted in such a way that if we split the set $V(\eta)$ into sets V_1, V_2, V_3, V_4 as done previously, then we have there is either a path of length at most $2k$ between some vertex in V_1 and some vertex in V_3, or a path of length at most $2k$ between some vertex in V_3 and some vertex in V_4. As a reflex, the value $8d$ in Lemma 13 and in all statements of Section 5.4 may be substituted with $8k$ when considering k-outerplanar graphs. ◀

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6 Proofs of Results from Section 4

Restatement of Proposition 1 . *If H^h is an η-respecting plane completion of G^g, then $H_\eta^{g_\eta}$ is a (G^g, η)-completion.*

Proof. Let H^h be a η-respecting plane completion fo G^g. Then by definition, η is also a H^h-noose. Additionally, the graph $H_\eta^{h_\eta}$ lies, also by definition, in the closed interior of η. Finally, since H^h is a plane supergraph of G^g then $H_\eta^{h_\eta}$ is a plane supergrpah of $G_\eta^{g_\eta}$. Therefore, $H_\eta^{h_\eta}$ is a (G^g, η)-completion. ◀

Restatement of Proposition 2 . *Let $|V(\eta)| \leq 8d$. Then there exist at most $2^{d^{O(d)}}$ distinct signatures.*

Proof. Clearly, there are at most $d^{O(d^2)}$ distinct matrices $D(X^x)$. On the other hand, for each unresolved pair u, u', the vector $D(X^x, u, u')$ can be picked among at most $d^{O(d)}$ choices. Finally, there are $2^{d^{O(d)}}$ possible subsets of such choices. ◀

Restatement of Lemma 4 . *Let H^h be a plane completion of G^g of diameter at most d and let X^x be a (G^g, η)-completion such that $\mathcal{S}(H_\eta^{h_\eta}, \eta) = \mathcal{S}(X^x, \eta)$. Then if H^g has diameter at most d, $(H^h - H_\eta^{h_\eta}) + X^x$ has diameter at most d.*

Proof. Assume that H^h has diameter at most d, and let $v, v' \in \hat{V}(\eta)$ be unresolved in X^x. Since $H_\eta^{h_\eta}$ and X^x have the same signature, there exist unresolved vertices u and u' in $H_\eta^{h_\eta}$ such that $D(H_\eta^{h_\eta}, u, u') = D(X^x, v, v')$. Since H^h has diameter at most d, there exists $r \leq d$, some vertices $u_0, u_1, ..., u_r$ and paths $p_1, p_2, ..., p_r$ such that the following conditions are satisfied.

1. $u_0 = u$, $u_r = u'$ and $\{u_2, ..., u_{r-1}\} \subseteq V(\eta)$.
2. For each $j \in [r]$, p_j is a path of minimum length from u_{j-1} to u_j.
3. For each $j \in [r]$, p_j the internal vertices of p_j are not in $V(\eta)$.

Now we let $p'_1, p'_2, ..., p'_r$ be paths in $(H^h - H_\eta^{h_\eta}) + X^x$, defined as follows for each $j \in \{1, ..., r\}$.

1. If the interior vertices of path p_j belong to $V(G) \backslash \hat{V}(\eta)$, then all vertices and edges of p_j belong to the graph $(H^h - H_\eta^{h_\eta}) + X^x$. Therefore, in this case, we set $p'_j = p_j$.
2. If both u_{j-1} and u_j belong to $V(\eta)$ and all interior vertices of p_j belong to $\hat{V}(\eta)$, then we let p'_j be any path of minimum length from u_{j-1} to u_j in the graph X^x. Note that such a path exists and has the same length as p_j due to the fact that X^x and $H_\eta^{h_\eta}$ have the same signature.
3. Now finally, let p'_1 be a path of length at most $l(p_1)$ from v and u_1 in X^x, and let p_r be a path of length at most $l(p_r)$ from u_{r-1} to v' in X^x. We observe that such paths exist, since X^x and $H_\eta^{g_\eta}$ have the same signature.

Therefore we have that $p'_1 p'_2 ... p'_r$ induces a walk in $(H^h - H_\eta^{g_\eta}) + X^x$ of length at most d. This shows that the distance between v and v' is at most d. ◀

7 Proof of Statements From Section 5

Restatement of Proposition 3 . *Let η be a G^g-noose, and let uv be an edge in $E(G)$ such that uv is internally parallel to η. Then there exists a G^g-noose η' such that $V(\eta) = V(\eta')$, uv is externally parallel to η' and $G_{\eta'}^{g_{\eta'}} = G_\eta^{g_\eta} - uv$.*

Proof. Let $\overline{F}$ and $\overline{F}'$ be the (topological) faces of G^g that contain the simple arc $g(uv)$. Since uv is internally parallel to η, there exists a simple arc ℓ satisfying conditions 1-3 defining the notion of parallel edge. Assume that $\ell \subseteq \overline{F}$ and let $m = \eta \backslash Int(\ell)$. Let ℓ' be a simple arc contained in $\overline{F}'$ such that $\ell' \cap g(V(G)) = \{g(u), g(v)\}$ and $\ell' \cap g(u'v') \subseteq \{g(u), g(v)\}$ for every $u'v' \in E(G)$ with $u' \neq u$ or $v' \neq v$. Finally, we set $\eta' = m \cup \ell'$. In other words, η' is obtained by deleting the arc ℓ out of η and by gluing the arc ℓ' in its place. Then it should be clear that $V(\eta) = V(\eta')$, uv is externally parallel to η' and $G_{\eta'}^g = G_\eta^g - uv$. ◀

Restatement of Lemma 7 . *Let uv be an edge in $E(G)$, and let η be a G^g-noose such that uv is internally parallel to η. Let η' be a G^g-noose such that $V(\eta) = V(\eta')$, $G_{\eta'}^{g_{\eta'}} = G_\eta^{g_\eta} - uv$ and uv is externally parallel to η'. Note that such a noose η' exists by Proposition 3. Suppose that $X^x \in \mathcal{F}_\eta$. Then there exists $Y^y \in \mathcal{F}_{\eta'}$ such that $Y^y = X^x - uv$.*

Proof. Let H^h be a η-respecting diameter-d plane completion of G^g, and let $H_\eta^{h_\eta} = X^x \in \mathcal{F}_\eta$ be the plane graph obtained by restricting H^h to the closed interior of η. Note that since the edge uv is internally parallel to η in G^g, we have that uv is also internally parallel to η in H^h. By Proposition 3, there exists an H^h-noose η' such that $H_{\eta'}^{h_{\eta'}} = H_\eta^{h_\eta} - uv$. Now let $Y^y = H_{\eta'}^{h_{\eta'}}$ then by definition of $\mathcal{F}_{\eta'}$, $Y^y \in \mathcal{F}_{\eta'}$. Additionally, since by assumption $X^x = H_\eta^{g_\eta}$, we have that $Y^y = X^x - uv$. ◀

Restatement of Lemma 8 . *Let uv be an edge in $E(G)$, and let η and η' be G^g-nooses such that $V(\eta) = V(\eta')$ and $G_\eta^g = G_{\eta'}^g + uv$. Assume that $\tilde{\mathcal{F}}_{\eta'}$ represents $\mathcal{F}_{\eta'}$. Then $\tilde{\mathcal{F}}_\eta = \tilde{\mathcal{F}}_{\eta'} + uv$ represents $\mathcal{F}_\eta$.*

Proof. Let H^h be a diameter-d plane completion of G^g, and let $H_\eta^h = X^x \in \mathcal{F}_\eta$ be its restriction to the closed interior of η. Then by Lemma 7 there is some $Y^y \in \mathcal{F}_{\eta'}$ such that $Y^y = X^x - uv$. Now, by assumption, there exists some $\tilde{Y}^{\tilde{y}} \in \tilde{\mathcal{F}}_{\eta'}$ such that $(H^h - Y^y) + \tilde{Y}^{\tilde{y}}$ is also a diameter-d plane completion of G^g. Now let $\tilde{X}^{\tilde{x}} = \tilde{Y}^{\tilde{y}} + uv$. Then we have that $(H^h - (X^x - uv)) + (\tilde{X}^{\tilde{x}} - uv)$ is also a diameter-d plane completion of G^g. But the following equivalence can be verified.

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$$(H^h - (X^x - uv)) + (\tilde{X}^{\tilde{x}} - uv) \equiv (H^h - X^x) + \tilde{X}^{\tilde{x}}$$

Intuitively, in the left-hand side, the edge uv is never removed or added to H^h. On the other hand, in the right-hand side, the edge uv is first removed together with X^x and subsequently this edge is added back together with $\tilde{X}^{\tilde{x}}$. Therefore, we have that $(H^h - X^x) + \tilde{X}^{\tilde{x}}$ is a diameter-d plane completion of G^g.

In other words, by assuming H^h is a diameter-d plane completion of G^g and that $H^h_\eta = X^x \in \mathcal{F}_\eta$, we have concluded that there exists an $\tilde{X}^{\tilde{x}} \in \tilde{\mathcal{F}}_\eta$ such that $(H^h_\eta - X^x) + \tilde{X}^{\tilde{x}}$ is also a diameter-d plane completion of G^g. This shows that $\tilde{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$.

◀

Restatement of Lemma 9 . *Let η be an extensible noose. Suppose that $\tilde{\mathcal{F}}_{\eta'}$ represents $\mathcal{F}_{\eta'}$ for every $\eta' \in \mathrm{Ext}(\eta)$. Then $\hat{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$.*

Proof. Let H^h be a diameter-d plane completion of G^g. Then by Equation 2, there is a trivial extension η' of η such that $H^h \in \mathcal{F}_{\eta'}$. Since by assumption, $\tilde{\mathcal{F}}_{\eta'}$ represents $\mathcal{F}_{\eta'}$, and since by definition $\tilde{\mathcal{F}}_{\eta'} \subseteq \hat{\mathcal{F}}_\eta$, we have that $\mathcal{F}_\eta$ is represented by $\hat{\mathcal{F}}_\eta$. ◀

Restatement of Lemma 10 . *Let η be an extensible noose and let $\tilde{\mathcal{F}}_\eta = Trunc(\hat{\mathcal{F}}_\eta)$. Then $\tilde{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$.*

Proof. By Lemma 9, $\hat{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$. By Lemma 5, $\tilde{\mathcal{F}}_\eta$ represents $\mathcal{F}_\eta$. ◀

Restatement of Lemma 11 . *There is an edge $uv \in E(H^{h_\eta}_\eta)$ such that either $u \in \hat{V}_1$ and $v \in \hat{V}_3$, or $u \in \hat{V}_2$ and $v \in \hat{V}_4$.*

Proof. Suppose for contradiction that no such an edge uv exists. Then there is a noose with no edge in the interior which has at least one vertex from each $\hat{V}_i$. But this contradicts the assumption that H^h is triangulated. ◀

Restatement of Lemma 12 . *At least one of the following statements must be satisfied.*

1. *There is a path of length at most $2d + 1$ between a vertex in V_1 to a vertex in V_3.*
2. *There is a path of length at most $2d + 1$ between a vertex in v_2 and a vertex in v_4.*

Proof. By Lemma 11, there exists an edge $uv \in E(H^{h_\eta}_\eta)$ such that either $u \in \hat{V}_1$ and $v \in \hat{V}_3$, or $u \in \hat{V}_2$ and $v \in \hat{V}_4$. We will show that the former case implies the existence of a path of length at most $2d + 1$ from a vertex $u' \in V_1$ to a vertex $v' \in V_3$. An analog argument, which we skip, shows that the latter case implies the existence of a path of length at most $2d + 1$ between a vertex in V_2 and a vertex in V_4.

Let $u' \in V_1$ be a vertex at minimum distance from u and let p_1 be a path of minimum length from u' to u. Analogously, let $v' \in V_2$ be a vertex at minimum distance from v, and let p_2 be a path of minimum length from v to v'. Then the path p_1p_2 is a path of from u' to v'. Since H^h has diameter at most d, both p_1 and p_2 have length at most d. Therefore, the path p_1p_2 has length at most $2d + 1$. ◀

Restatement of Lemma 13 . *Let H^h be a η-respecting diameter-d plane triangulated completion of G^g. Then there exist G^g-nooses η_1 and η_2 satisfying the following properties.*

1. *$|V(\eta_1)| \leq 8d$ and $|V(\eta_2)| \leq 8d$.*
2. *$\eta = \eta_1 \oplus \eta_2$.*
3. *H^h is both η_1-respecting and η_2-respecting.*
4. *$H^{h_\eta}_\eta = H^{h_{\eta_1}}_{\eta_1} \cup H^{h_{\eta_2}}_{\eta_2}$.*

Proof. For each $i \in \{1, ..., 4\}$, let $V_i = \{v_1^i, ..., v_{r_i}^i\}$. Where for each $i \in \{1, ..., 4\}$ and each $j \in [r_i - 1]$, v_j^i and v_{j+1}^i are consecutive in η, and $v_{r_1}^1$ and v_1^2 are consecutive, $v_{r_2}^2$ and v_1^3 are consecutive, $v_{r_3}^3$ and v_1^4 are consecutive, and $v_{r_4}^4$ and v_1^1 are consecutive. By Lemma 12, there exists an edge $uv \in H^h$ such that either $u = v_j^1$ for some $j \in \{1, ..., r_1\}$ and $v = v_{j'}^3$ for some $j' \in \{1, ..., r_3\}$, or $u = v_j^2$ for some $j \in \{1, ..., r_2\}$ and $v = v_{j'}^4$ for some $j' \in \{1, ..., r_4\}$. We will show that the former case implies that there exist G^g-nooses η_1 and η_2 which satisfies Conditions 1-4 above. The proof that the latter case also implies the existence of such G^g-nooses is analog.

Let $p = v_1 v_2 ... v_r$ for $r \leq d+1$ be a path from $v_1 = v_j^1$ to $v_r = v_{j'}^3$. Then we set η_1 to be the noose with vertex set

$$V_4 \cup \{v_1^1, ..., v_j^1\} \cup \{v_1^3, ..., v_{j'}^3\} \cup \{v_2, ..., v_r - 1\}$$

where every edge of p is externally parallel to η_1. Analogously, we let η_2 be the noose with vertex set

$$V_2 \cup \{v_j^1, ..., v_{r_1}^1\} \cup \{v_{j'}^3, ..., v_{r_3}^3\} \cup \{v_2, ..., v_{r-1}\}$$

where every edge of p is internally parallel to η_2. Then condition 1 is satisfied, since the size of the noose is at most $2d + j + j' + r - 1 \leq 8d$. By construction, $\eta = \eta_1 \oplus \eta_2$, and therefore condition 2 is satisfied. Since no edge of H^h crosses either noose η_1 or η_2, we have that H^h is both η_1-respecting and η_2-respecting. Finally by identifying the vertices in the vertices $v_1, ..., v_r$ in $H_{\eta_1}^{h_{\eta_1}}$ with the vertices $v_1, ..., v_r$ in $H_{\eta_2}^{h_{\eta_2}}$ we get the graph $H_\eta^{h_\eta}$ and therefore, condition 4 is satisfied. ◀

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