Kernelization and Sparseness: the case of Dominating Set*

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Abstract

The search for linear kernels for the Dominating Set problem on classes of graphs of a topological nature has been one of the leading trends in kernelization in recent years. Following the fundamental work of Alber et al. [2] that established a linear kernel for the problem on planar graphs, linear kernels have been given for bounded-genus graphs [4], apex-minor-free graphs [15], H-minor-free graphs [16], and H-topological-minor-free graphs [17]. These generalizations are based on bidimensionality and powerful decomposition theorems for H-minor-free graphs and H-topological-minor-free graphs of Robertson and Seymour [28] and of Grohe and Marx [22].

In this work we investigate a new approach to kernelization algorithms for Dominating Set on sparse graph classes. The approach is based on the theory of bounded expansion and nowhere dense graph classes, developed in the recent years by Nešetřil and Ossona de Mendez, among others. More precisely, we prove that Dominating Set admits a linear kernel on any hereditary graph class of bounded expansion and an almost linear kernel on any hereditary nowhere dense graph class. Since the class of H-topological-minor-free graphs has bounded expansion, our results strongly generalize all the above mentioned works on kernelization of Dominating Set. At the same time, our algorithms are based on relatively short and self-contained combinatorial arguments, and do not depend on bidimensionality or decomposition theorems.

Finally, we prove that for the closely related Connected Dominating Set problem, the existence of such kernelization algorithms is unlikely, even though the problem is known to admit a linear kernel on H-topological-minor-free graphs [17]. Thus, it seems that whereas for Dominating Set sparsity is enough to guarantee the existence of an efficient kernelization algorithm, for Connected Dominating Set stronger constraints of topological nature become necessary.

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1 Introduction

Domination and kernelization. In the classical Dominating Set problem, given a graph G and an integer k, we are asked to determine the existence of a subset $D \subseteq V(G)$ of size at most k such that every vertex $u \in V(G)$ is *dominated* by D: either u belongs to D itself, or it has a neighbor that belongs to D. The problem is NP-hard and remains so even in very restricted settings, e.g. on planar graphs of maximum degree 3 (cf. [GT2] in Garey and Johnson [20]). The complexity of Dominating Set was studied intensively under different algorithmic frameworks, most importantly from the points of view of approximation and of parameterized complexity. In this work we are interested in the latter paradigm.

Dominating Set parameterized by the target size k plays a central role in parameterized complexity as it is a predominant example of a W[2]-complete problem. Recall that the main focus in parameterized complexity is on designing *fixed-parameter* algorithms, or shortly *FPT* algorithms, whose running time on an instance of size n and parameter k has to be bounded by $f(k) \cdot n^c$ for some computable function f and constant c. Downey and Fellows introduced a hierarchy of parameterized complexity classes $\mathrm{FPT} \subseteq \mathrm{W}[1] \subseteq \mathrm{W}[2] \subseteq \ldots$ that is believed to be strict, see [9,14]. As Dominating Set is W[2]-complete in general, we do not expect it to be solvable in FPT time.

However, it turns out that various restrictions on the input graph lead to robust tractability of Dominating Set. Out of these, a particularly fruitful line of research concerned investigation of the complexity of the problem in sparse graph classes, like planar graphs, graphs of bounded genus, or graphs excluding some fixed graph H as a minor. In these classes we can even go one step further than just showing fixed-parameter tractability; It is possible to design a linear kernel for the problem: Formally, a *kernelization algorithm* (or a *kernel*) is a polynomial-time preprocessing procedure that given an instance (I, k) of a parameterized problem outputs another instance (I', k') of the same problem which is equivalent to (I, k), but whose total size $|I'| + k'$ is bounded by $f(k)$ for some computable function f, called the *size* of the kernel. If f is polynomial (resp. linear), then such an algorithm is called a *polynomial* (resp. *linear*) *kernel*. Note that the existence of such a kernelization algorithm immediately implies that the problem can be solved by a very efficient fixed-parameter algorithm: after applying kernelization, any brute-force search or more clever algorithm runs in time bounded by a function of k only.

The quest for small kernels for Dominating Set on sparse graph classes began with the groundbreaking work of Alber et al. [2], who showed the first linear kernel for the problem on planar graphs. This work also introduced the concept of a *region decomposition*, which proved to be a crucial tool for constructing linear kernels for other problems on planar graphs later on. Another important step was the work of Alon and Gutner [3,23], who gave an $\mathcal{O}(k^c)$ kernel for the problem on H-topological-minor free graphs, where c depends on H only. Moreover, if $H = K_{3,h}$ for some h, then the size of the kernel is actually linear. This led Alon and Gutner to pose the following excellent question: Can one characterize the families of graphs where Dominating Set admits a linear kernel?

The research program sketched by the works of Alber et al. [2] and Alon and Gutner [3,23] turned out to be one of particularly fruitful directions in parameterized complexity in recent years, and eventually led to the discovery of new and deep techniques. In particular, linear kernels for Dominating Set have been given for bounded genus graphs [4], apex-minor-free graphs [15], H-minor-free graphs [16], and most recently H-topological-minor-free graphs [17]. In all these results, the notion of *bidimensionality* plays the central role. Using variants of the Grid Minor Theorem, it is possible to understand well the connections between the minimum possible size of a dominating set in a graph and its treewidth. The considered graph classes also admit powerful decomposition theorems that follow from the Graph Minors project of Robertson and Seymour [28], or the recent work of Grohe and Marx [22] on excluding H as a topological minor. The combination of these tools provides a robust base for a structural analysis of the input instance, which leads to identifying *protrusions*: large portions of the graph that have constant treewidth and small interaction with other vertices, and hence can be efficiently replaced by smaller gadgets. The protrusion approach, while originating essentially in the work on the Dominating Set problem, turned out to be a versatile tool for finding efficient preprocessing routines for a much wider class of problems. In particular, the *meta-kernelization* framework of Bodlaender et al. [4], further refined by Fomin et al. [15], describes how a combination of bidimensional and finite-state properties of a generic problem leads to the construction of linear kernels on bounded genus and H-minor-free graphs.

Beyond the current frontier of H-topological-minor-free graphs [17], kernelization of Dominating Set was studied in graphs of bounded degeneracy. Recall that a graph is called *d-degenerate* if every subgraph contains a vertex of degree at most d. Philip et al. [27] obtained a kernel of size $\mathcal{O}(k^{(d+1)^2})$ on d-degenerate graphs for constant d, and more generally a kernel of size $\mathcal{O}(k^{\max(i^2,j^2)})$ on graphs excluding the complete bipartite graph $K_{i,j}$ as a subgraph. However, as proved by Cygan et al. [5], the exponent of the size of the kernel needs to increase with d at least quadratically: the existence of an $\mathcal{O}(k^{(d-1)(d-3)-\varepsilon})$ kernel for any $\varepsilon > 0$ would imply that NP $\subseteq$ coNP/poly. Thus, in these classes the existence of a linear kernel is unlikely.

Sparsity. The concept of sparsity has been recently the subject of intensive study both from the point of view of pure graph theory and of computer science. In particular, the notions of graph classes of *bounded expansion* and *nowhere dense* graph classes have been introduced by Nešetřil and Ossona de Mendez. The main idea behind these models is to establish an abstract notion of sparsity based on known properties of well-studied sparse graph classes, e.g. H-minor-free graphs, and to develop tools for combinatorial analysis of sparse graphs based only on this abstract notion. We refer to the book of Nešetřil and Ossona de Mendez [26] for an introduction to the topic.

Intuitively, a graph class $\mathcal{G}$ has *bounded expansion* if any minor obtained by contracting disjoint subgraphs of radius at most r is d_r-degenerate, for some constant d_r. Thus, this property can be thought of as strengthened degeneracy that persists after very constrained minor operations. The notion of a *nowhere dense* graph class is a further relaxation of this concept; we refer to Definition 2.13 for a formal definition. In particular, every graph class $\mathcal{G}$ that has bounded expansion is also nowhere dense, and all the aforementioned classes on which the existence of a linear kernel for Dominating Set has been established (planar, bounded genus, H-minor-free, H-topological-minor-free) have bounded expansion.

From the point of view of theoretical computer science, of particular importance is the program of establishing fixed-parameter tractability of model checking first order logic on sparse graphs. A long line of work resulted in FPT algorithms for model checking first order formulas on more and more general classes of sparse graphs [7,11,13,18,21,29], similarly to the story of kernelization of Dominating Set. Finally, FPT algorithms for the problem have been given for graph classes of bounded expansion by Dvořák et al. [11], and very recently for nowhere dense graph classes by Grohe et al. [21]. This is the ultimate limit of this program: as proven in [11], for any class $\mathcal{G}$ that is not nowhere dense (is *somewhere dense*) and is closed under taking subgraphs, model checking first order formulas on $\mathcal{G}$ is not fixed-parameter tractable (unless FPT = W[1]).

Fixed-parameter tractability of Dominating Set on nowhere dense graph classes follows immediately from the result of Grohe et al. [21], since the problem is definable in first order logic. However, an explicit algorithm was given earlier by Dawar and Kreutzer [8].

To summarize, we would like to stress that Dominating Set has repeatedly served as a trigger for developing new techniques in parameterized complexity: the subexponential algorithm on planar graphs [1] lead to the theory of bidimensionality; the kernelization algorithm on planar graphs [2] initiated meta-theorems and protrusion-based techniques on planar graphs and beyond, which were further refined by techniques developed for graphs with excluded topological minor [17]; and, last but not least, the work on Dominating Set in nowhere-dense graphs [8] lead to generic first order logic results on sparse classes of graphs. Therefore, we believe that understanding the kernelization status of Dominating Set in sparse graph classes may again lead to very fruitful developments.

Our results. In this work we prove that having bounded expansion or being nowhere dense is sufficient for a graph class to admit an (almost) linear kernel for Dominating Set. Henceforth, for a graph G, we let $\mathbf{ds}(G)$ denote the size of a minimum dominating set of G.

Theorem 1.1. *Let $\mathcal{G}$ be a graph class of bounded expansion. There exists a polynomial-time algorithm that given a graph $G \in \mathcal{G}$ and an integer k, either correctly concludes that $\mathbf{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size $\mathcal{O}(k)$ with the property that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$.*

Theorem 1.2. *Let $\mathcal{G}$ be a nowhere dense graph class and let $\varepsilon > 0$ be a real number. There exists a polynomial-time algorithm that given a graph $G \in \mathcal{G}$ and an integer k, either correctly concludes that $\mathbf{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size $\mathcal{O}(k^{1+\varepsilon})$ with the property that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$.*

In both cases, to obtain a kernel we apply the algorithm given by Theorem 1.1 or 1.2, and then either provide a trivial no-instance (in case the algorithm concluded that $\mathbf{ds}(G) > k$), or we output $(G[Y], k)$. This immediately yields the following:

Corollary 1.3. *For every hereditary graph class $\mathcal{G}$ with bounded expansion,* Dominating Set *admits a kernel of size $\mathcal{O}(k)$ on graphs from $\mathcal{G}$. For every hereditary and nowhere dense graph class $\mathcal{G}$ and every $\varepsilon > 0$,* Dominating Set *admits a kernel of size $\mathcal{O}(k^{1+\varepsilon})$ on graphs from $\mathcal{G}$.*

Note that we formally need to assume that the graph class $\mathcal{G}$ is hereditary (closed under taking induced subgraphs), in order to ensure that the output instance $(G[Y], k)$ is of the same problem as the input one. However, this is a purely formal problem: for any class $\mathcal{G}$ that either has bounded expansion or is nowhere dense, its closure under taking induced subgraphs also has this property, with exactly the same expansion parameters. So for the sake of kernelization we can always remain in the closure of $\mathcal{G}$ under taking induced subgraphs.

The obtained results strongly generalize the previous results on linear kernels for Dominating Set on sparse graph classes [2,4,15–17], since all the graph classes considered in these results have bounded expansion. However, we see the main strength of our results in that they constitute an abrupt turn in the current approach to kernelization of Dominating Set on sparse graphs: the tools used to develop the new algorithms are radically different from all the previously applied techniques. Instead of investigating bidimensionality and treewidth, and relying on intricate decomposition theorems originating in the work on graph minors, our algorithms exploit only basic properties of bounded expansion and nowhere dense graph classes. As a result, this paper presents essentially self-contained proofs of Theorems 1.1 and 1.2 that rely on simple combinatorial arguments and span over just a few pages. The only external facts that we use are basic properties of weak colorings and the constant-factor approximation algorithm for Dominating Set of Dvořák [10]. All in all, the results show that only the combinatorial sparsity of a graph class is essential for designing (almost) linear kernels for Dominating Set, and further topological constraints like excluding some (topological) minor are unnecessary.

We complement our study by proving that for the closely related Connected Dominating Set problem, where the sought dominating set D is additionally required to induce a connected subgraph, the existence of even polynomial kernels for bounded expansion and nowhere dense graph classes is unlikely. More precisely, we prove the following result:

Theorem 1.4. *There exists a class of graphs $\mathcal{G}$ of bounded expansion such that* Connected Dominating Set *does not admit a polynomial kernel when restricted to $\mathcal{G}$, unless* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$*, and furthermore, $\mathcal{G}$ is closed under taking subgraphs.*

Up to this point, linear kernels for Connected Dominating Set were given for the same family of sparse graph classes as for Dominating Set: a linear kernel for the problem on H-topological-minor-free graphs was obtained by Fomin et al. [17]. Hence, classes of bounded expansion constitute the point where the kernelization complexity of both problems diverge: while Dominating Set admits a linear kernel by Theorem 1.1, for Connected Dominating Set even a polynomial kernel is unlikely by Theorem 1.4. We find this difference very surprising, and our intuition about the phenomenon is as follows: the connectivity constraint has a completely different nature, and topological properties of the graph class become necessary to handle it efficiently. Indeed, a deeper examination of the proof of Theorem 1.4 shows that we essentially exploit only the connectivity constraint to establish the lower bound.

Our techniques. As explained before, the techniques applied to prove Theorems 1.1 and 1.2 differ radically from tools used in the previous works [2,4,15–17]. The main reason is that so far all the approaches were based on bidimensionality and decomposition theorems for graph classes with topological constraints, like H-(topological)-minor-free graphs. For bounded expansion and nowhere dense graphs, there are no known decomposition theorems. Bidimensional arguments also cease to work, since they are inextricably linked to surface embeddings of graphs, meaningless in the world of nowhere dense and bounded expansion graph classes.

The failure of known techniques, seemingly a large obstacle for our project, actually came as a blessing as it forced us to search for the "real" reasons why Dominating Set admits linear kernels on sparse graph classes; Identifying the right tools enabled us to streamline the reasoning so that it is significantly simpler than the previous works. We now briefly describe the main approach on the example of Theorem 1.1. The proof of Theorem 1.2 uses exactly the same approach, with the difference that some arguments for graphs of bounded expansion need to be replaced with analogous results for nowhere dense graphs.

The first idea is to kernelize the instance in two phases: Intuitively, in the first phase we reduce the number of *dominatees*, vertices whose domination is essential, and in the second phase we reduce the number of *dominators*, vertices that are sensible to use to dominate other vertices. In order to formalize this approach, we introduce the notion of a domination core: a subset $Z \subseteq V(G)$ is a *domination core* if every minimum-size subset $D \subseteq V(G)$ that dominates Z is guaranteed to dominate the whole graph. Hence, every vertex whose domination is identified as irrelevant can safely be removed from the domination core. In the first phase of the algorithm we find a domination core in the graph of size linear in the parameter k, and in the second phase we reduce the number of vertices outside it. The first phase is the most difficult one, while the second is much simpler.

The small domination core is found iteratively, by first taking $Z = V(G)$ and then removing vertices from Z one by one. Hence, the main difficulty is to find a vertex that can be safely removed from Z; for simplicity, we focus on the first iteration when $Z = V(G)$. The first step is to apply the approximation algorithm of Dvořák for Dominating Set on graphs of bounded expansion [10]. This algorithm has the following very important feature: given a parameter k, it either provides a dominating set of size $\mathcal{O}(k)$, or it outputs a proof that $\mathbf{ds}(G) > k$ in the form of a 2-*scattered set* S of size larger than k where S is 2-scattered if every two vertices of S are at distance more than 2 from each other. The idea is to apply the algorithm of Dvořák repeatedly: In each iteration we either identify another approximate dominating set and remove it from the graph, or we find a large 2-scattered set in the remaining instance and terminate the iteration. As we work in a graph of bounded expansion, it can be shown that this process terminates after a constant number of steps. Hence, we end up with the following structure in the graph: a dominating set $X \subseteq V(G)$ of size $\mathcal{O}(k)$, and a set $S \subseteq V(G) \setminus X$ that is 2-scattered in $G - X$. By carefully selecting the parameters of the approximation, we can ensure that $|S| > c|X|$ for as large a constant c as we like.

Having identified such a pair (X, S), we partition $V(G) \setminus X$ into equivalence classes such that two vertices are equivalent when they have exactly the same neighborhood in X. The vertices of S are handled slightly differently: before the partitioning we contract their whole neighborhoods in $G - X$ onto them. Now is the moment where we crucially use the fact that the graph under consideration has bounded expansion: it can be shown that the number of equivalence classes is linear in the size of X. This neighborhood diversity argument (formally, Proposition 2.5) appears again and again in our proofs, and in our opinion it is the main reason why linear kernelization of Dominating Set on sparse graph classes is possible. We further strengthen this argument (see Lemma 2.10) to prove that not only are there few classes, but their mutual interaction forms a graph of bounded expansion. Based on these observations, we can use the potential method to identify a class κ, where the number of vertices from S is large compared to the number of other classes with which κ interacts. We then show that an appropriately chosen member of κ is an irrelevant dominatee that can be removed from Z.

This reasoning can be applied as long as $|Z| > Ck$ for some constant C, so we eventually compute a domination core of size linear in k. To remove the dominators, we again apply the neighborhood diversity argument. We partition the vertices of $V(G) \setminus Z$ into classes with respect to their neighborhoods in Z, the number of these classes is linear in $|Z|$, and it is safe to remove all but one vertex from each class.

The proof of Theorem 1.4 uses the technique of compositionality to refute the existence of a polynomial kernel, and is based on the kernelization hardness result for Connected Dominating Set on 2-degenerate graphs presented by Cygan et al. [6]. The output instances of the original construction of Cygan et al. [6] do not have bounded expansion, but after adding a number of new technical ideas the construction can be modified to ensure this property.

Organization of the paper. In Section 2 we recall the most important definitions and facts about bounded expansion and nowhere dense classes of graphs, as well as prove some auxiliary results that will be used later on. Section 3 contains the proof of Theorem 1.1 — the main result for bounded expansion classes — whereas Section 4 contains the proof of Theorem 1.2 — the main result for nowhere dense classes. In Section 5 we present the lower bound for Connected Dominating Set, i.e., Theorem 1.4. Section 6 contains concluding remarks and prospects for future work. Proofs of auxiliary facts (marked with $\star$) that are very easy and/or follow directly from known results have been deferred to Appendix 6 in order not to distract the attention of the reader.

2 Preliminaries

2.1 Notation

Basic graph notation All graphs we consider are finite, simple, and undirected. For a graph G, we denote by $|G| = |V(G)|$ the number of vertices and by $||G|| = |E(G)|$ the number of edges in G. The *density* of a graph G, denoted density(G) is defined as density$(G) = ||G||/|G|$. For an integer $k \in \mathbb{N}$ we denote by $[k] = \{1, \ldots, k\}$ the first k positive integers.

For a vertex v in a graph G, we denote by $N_G(v) = \{u \colon uv \in E(G)\}$ the *open neighborhood* of v and by $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v in G. These notions can be naturally extended to sets of vertices $X \subseteq V(G)$ as follows: $N_G[X] = \bigcup_{v \in X} N_G[v]$ and $N_G(X) = N_G[X] \setminus X$. If G is clear from the context, we omit the subscripts. Furthermore, we write $N_X(v)$ to denote the neighborhood of v restricted to X, i.e., $N_X(v) = N_G(v) \cap X$, and refer to it as the X-neighborhood of v. The *degree* of a vertex $v \in V(G)$ is the number of neighbors it has, i.e., $\deg(v) = |N(v)|$.

The induced subgraph $G[X]$ for $X \subseteq V(G)$ is the graph with vertex set X and for $x_1, x_2 \in X$ we have that $x_1x_2 \in E(G[X])$ if and only if $x_1x_2 \in E(G)$. A graph $H = (V_H, E_H)$ is a subgraph of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E(G[V_H])$. We will say that H is a subgraph of G if H is isomorphic to a subgraph of G. For a set of vertices $X \subseteq V(G)$, we write $G - X$ to denote the induced subgraph $G[V(G) \setminus X]$.

Given a graph G and two vertex subsets $D, Z \subseteq V(G)$, we say that D is a *Z-dominator* if D dominates Z in G, that is, every vertex $z \in Z \setminus D$ has a neighbor in D. We denote by $\mathbf{ds}(G, Z)$ the size of a smallest Z-dominator of G. By $\mathbf{ds}(G)$ we mean $\mathbf{ds}(G, V(G))$, i.e., the size of a smallest dominating set in G. A set $S \subseteq V(G)$ is ℓ-scattered in G if for every pair of distinct vertices $s_1, s_2 \in S$, the distance between s_1 and s_2 is at least $\ell + 1$, i.e., any path from v_1 to v_2 has at least ℓ internal vertices. Note that if there is a 2-scattered set S of size k, then any dominating set of G must have size at least k, since every vertex of G can dominate at most one vertex of S. Hence, we call a 2-scattered set of size $k + 1$ an obstruction for a dominating set of size k.

A *clique* in a graph is a subset of pairwise adjacent vertices. We write $\omega(G)$ to denote the *clique number* of a graph G, i.e., the size of a maximum clique in G. We write $\#\omega(G)$ to be the total number of cliques in G. By K_c we denote the complete graph on c vertices, and by K_{c_1,c_2} we denote the complete bipartite graph with the sides of the bipartition of sizes c_1 and c_2, respectively.

The *radius* of a graph G, denoted $\text{radius}(G)$ is the minimum integer r for which there exists a vertex $v \in V(G)$ (a *center*) such that every vertex in $V(G)$ is within distance at most r from v.

Minors and minor operations For an edge $e = uv$ in a graph G, the graph G/e is the graph obtained from *contracting* e, i.e., we replace the vertices u and v with a vertex w_{uv} that is adjacent to every vertex of $N_G(\{u, v\})$ in G/e. If $S \subseteq V(G)$ is a set of vertices such that $G[S]$ is connected, we let G/S denote the graph obtained from G by contracting S to a single vertex. That is, G/S is the graph obtained from deleting S from G and adding a vertex v_S which is adjacent to every vertex of $N_G(S)$; note that this is equivalent to contracting all the edges of any spanning tree of $G[S]$.

The reverse operation of contraction is the operation of *subdivision*. Given a graph G and an edge $uv = e \in E(G)$, the graph obtained from subdividing e in G is the graph with vertex set $V(G) \cup \{w_e\}$ and edge set $E(G) \setminus \{e\} \cup \{uw_e, vw_e\}$.

A graph H which is obtained from a graph G after a sequence of contractions is called a *contraction of* G. If H is subgraph of a contraction of G, then we say that H is a *minor* of G. A graph G is said to be H-minor-free if H is not a minor of G, and a graph class $\mathcal{G}$ is H-minor-free if every graph of $\mathcal{G}$ is H-minor-free.

2.2 Shallow minors, grad and expansion

Definition 2.1 (Shallow minor)**.** A graph M is an *r-shallow minor* of G, where r is an integer, if there exists a set of disjoint subsets $V_1, \ldots, V_{|M|}$ of $V(G)$ such that

1. each graph $G[V_i]$ is connected and has radius at most r, and
2. there is a bijection $\psi\colon V(M) \to \{V_1, \ldots, V_{|M|}\}$ such that for every edge $uv \in E(M)$ there is an edge in G with one endpoint in $\psi(u)$ and second in $\psi(v)$.

The set of all r-shallow minors of a graph G is denoted by $G \mathbin{\triangledown} r$. Similarly, the set of all r-shallow minors of all the members of a graph class $\mathcal{G}$ is denoted by $\mathcal{G} \mathbin{\triangledown} r = \bigcup_{G \in \mathcal{G}} (G \mathbin{\triangledown} r)$.

Definition 2.2 (Grad and bounded expansion)**.** For a graph G and an integer $r \geqslant 0$, we define the *greatest reduced average density (grad) at depth* r as

$$\nabla_r(G) = \max_{M \in G \,\triangledown\, r} \text{density}(M) = \max_{M \in G \,\triangledown\, r} ||M||/|M|.$$

We extend this notation to graph classes as $\nabla_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} \nabla_r(G)$. A graph class $\mathcal{G}$ then has *bounded expansion* if there exists a function $f \colon \mathbb{N} \to \mathbb{R}$ such that for all r we have that $\nabla_r(\mathcal{G}) \leqslant f(r)$.

Graph classes excluding a topological minor, such as planar and bounded-degree graphs, have bounded expansion [26]. Observe that bounded expansion implies bounded degeneracy, since $2\nabla_0(G)$ is equal to the degeneracy of G. However, the reverse does not hold: For an example, consider the class of cliques with each edge subdivided once.

Observe that for every graph G and integers $r \leqslant r'$, it holds that $\nabla_r(G) \leqslant \nabla_{r'}(G)$, and the same inequality holds for classes of graphs. Let us revisit some basic properties of grads that will be used later on.

Lemma 2.3. [$\star$] *Let G be a graph and let G' be obtained from G by adding a universal vertex to G, i.e., a vertex that is adjacent to every vertex of $V(G)$. Then*

$$\nabla_r(G) \leqslant \nabla_r(G') \leqslant \nabla_r(G) + 1.$$

The following proposition follows directly from the definition of grads.

Proposition 2.4. *For every graph class $\mathcal{G}$ and every pair of nonnegative integers r, s, the following holds: $(\mathcal{G} \,\triangledown\, s) \,\triangledown\, r \subseteq \mathcal{G} \,\triangledown (2rs + r + s)$. Consequently, $\nabla_s(G') \leqslant \nabla_{2rs+r+s}(\mathcal{G})$ for every $G' \in \mathcal{G} \,\triangledown\, r$. In particular, $\nabla_r(G') \leqslant \nabla_{3r+1}(\mathcal{G})$ and $\nabla_1(G') \leqslant \nabla_4(\mathcal{G})$ for each $G' \in \mathcal{G} \,\triangledown\, 1$.*

The following lemma about graphs of bounded expansion will be our main tool for constructing the kernel. It establishes a bound on the number of X-neighborhoods in a graph G, i.e., the number of subsets of X that are X-neighborhoods of some vertices outside X.

Proposition 2.5 ([19])**.** *Let G be a graph, $X \subseteq V(G)$ be a vertex subset, and $R = V(G) \setminus X$. Then for every integer $p \geqslant \nabla_1(G)$ it holds that*

1. $|\{v \in R \colon |N_X(v)| \geqslant 2p\}| \leqslant 2p \cdot |X|$*, and*
2. $|\{A \subseteq X \colon |A| < 2p \text{ and } \exists_{v \in R}\ A = N_X(v)\}| \leqslant (4^p + 2p)|X|$.

Consequently, the following bound holds:

$$|\{A \subseteq X \colon \exists_{v \in R}\ A = N_X(v)\}| \leqslant \left(4^{\nabla_1(G)} + 4\nabla_1(G)\right) \cdot |X|.$$

We most often apply Proposition 2.5 to a graph G belonging to some graph class $\mathcal{G}$ of bounded expansion. In all cases we will assume that $\nabla_1(\mathcal{G}) \geqslant 1$, and hence we will use a simpler form of the inequality:

$$|\{A \subseteq X \colon \exists_{v \in R}\ A = N_X(v)\}| \leqslant \left(4^{\nabla_1(G)} + 4\nabla_1(G)\right) \cdot |X| \leqslant \left(4^{\nabla_1(\mathcal{G})} + 4\nabla_1(\mathcal{G})\right) \cdot |X| \leqslant 2 \cdot 4^{\nabla_1(\mathcal{G})} \cdot |X|.$$

Another important property of graphs of bounded expansion is their stability under taking lexicographic products.

Definition 2.6 (Lexicographic product)**.** Given two graphs G and H, the *lexicographic product* $G \bullet H$ is defined as the graph on the vertex set $V(G) \times V(H)$ where the vertices (u, a) and (v, b) are adjacent if $uv \in E(G)$ or if $u = v$ and $ab \in E(H)$.

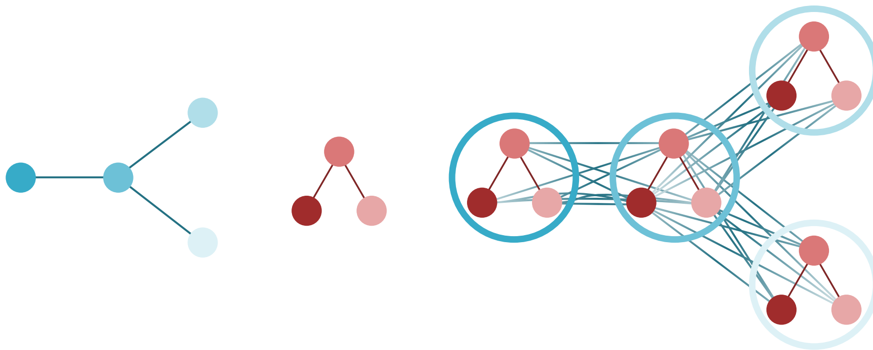

Figure 1: The lexicographic product of a claw and a P_3.

Figure 1 exemplifies this procedure. The following lemma shows that the grad of the lexicographic product of a graph and a complete graph is bounded.

Lemma 2.7. [$\star$] *For any graph G and non-negative integers $c \geqslant 1$ and r we have that*

$$\nabla_r(G \bullet K_c) \leqslant 4(8c(r+c) \cdot \nabla_r(G) + 4c)^{(r+1)^2}.$$

2.3 Weak colorings

Let $\Pi(G)$ denote the set of all linear orderings of $V(G)$. Given a graph G, an integer r and an ordering $\sigma \in \Pi(G)$, we say that a vertex u is *weakly r-accessible* from a vertex v in σ if $u <_\sigma v$ and there is a path P of length at most r with endpoints u and v such that every internal vertex w on P has the property that $u <_\sigma w$. We denote by $B_r^{G,\sigma}(v)$ the set of vertices that are weakly r-accessible from v in σ. When G is clear from context, we drop it from the superscript and write $B_r^\sigma(v)$.

Definition 2.8 (Weak r-coloring number)**.** We then define the *weak r-coloring* number of a graph G to be

$$\mathrm{wcol}_r(G) = 1 + \min_{\sigma \in \Pi(G)} \max_{v \in V(G)} |B_r^\sigma(v)|.$$

The weak coloring number of a graph is related to its grads, and in the following sections we need the following upper bound, which follows from [26, Proposition 4.8 and Theorem 7.11]:

Lemma 2.9 ([26])**.** *For any graph G, it holds that* $\mathrm{wcol}_2(G) \leqslant (8\nabla_1(G)^3 + 1)^2$.

2.4 Charging Lemma

Using the weak coloring number and its links with the grads, we can now strengthen the insight of Lemma 2.5 by showing that not only is the number of X-neighborhoods in a graph of bounded expansion small, but these X-neighborhoods are, in a sense, also uniformly spread on X. The following technical result is one of the most useful new contributions of this paper.

Lemma 2.10 (Charging Lemma)**.** *Let G be a bipartite graph with bipartition (X, Y) that belongs to some graph class $\mathcal{G}$ such that $\nabla_1(\mathcal{G}) \geqslant 1$. Suppose further that for every $u \in Y$ we have that $N(u) \neq \emptyset$, and that for every distinct $u_1, u_2 \in Y$ we have that $N(u_1) \neq N(u_2)$, i.e., Y is twin-free. Then there exists a mapping $\phi \colon Y \to X$ with the following properties:*

- *$u\phi(u) \in E$ for each $u \in Y$;*

- $|\phi^{-1}(v)| \leqslant 2^9 \cdot 4^{\nabla_1(G)} \cdot \nabla_1(G)^6 \leqslant 2^9 \cdot 4^{\nabla_1(\mathcal{G})} \cdot \nabla_1(\mathcal{G})^6$ *for each* $v \in X$.

In other words, there exists an assignment of vertices of Y to vertices of X such that each vertex of Y is assigned to one of its neighbors, and each vertex of X is charged by at most $f(\nabla_1(G))$ neighbors, for some function $f(\cdot)$.

Proof of Lemma 2.10. Since by Lemma 2.9 there exists an ordering σ of $V(G)$ such that for every vertex v, the set of weakly 2-accessible vertices $B_2^\sigma(v)$ has size at most $(8\nabla_1(\mathcal{G})^3+1)^2$.

Construct ϕ as follows: for every $u \in Y$, set $\phi(u)$ to that vertex of $N(u)$ that is last in σ; note that the validity of this definition is asserted by the assumption that Y does not contain isolated vertices. The first condition is trivially satisfied by ϕ, so we proceed to proving the second one.

To see that no vertex is charged too many times by ϕ, fix a vertex $v \in X$ and consider any vertex u with $\phi(u) = v$. First, suppose that $u <_\sigma v$: Then u is contained in $B_1^\sigma(v)$. Hence, the number of such vertices u is at most $|B_1^\sigma(v)|$.

Otherwise, i.e. when $v <_\sigma u$, note that by the definition of ϕ we have that $N(u) \setminus \{v\}$ lies entirely to the left of v. We infer that $N(u) \subseteq B_2^\sigma(v) \cup \{v\}$. Since Y is twin-free, we can immediately bound the number of such vertices u by $2^{|B_2^\sigma(v)|}$. However, we can obtain a better bound by applying Proposition 2.5 to $B_2^\sigma(v)$: the number of such vertices u is then bounded by

$$2 \cdot 4^{\nabla_1(\mathcal{G})} \cdot |B_2^\sigma(v)|.$$

Combining these two cases we obtain that

$$\begin{aligned}
|\phi^{-1}(v)| &\leqslant |B_1^\sigma(v)| + 2 \cdot 4^{\nabla_1(\mathcal{G})} \cdot |B_2^\sigma(v)| \\
&\leqslant 4 \cdot 4^{\nabla_1(\mathcal{G})} \cdot (8\nabla_1(\mathcal{G})^3+1)^2 \\
&\leqslant 4 \cdot 4^{\nabla_1(\mathcal{G})} \cdot 2^7 \nabla_1(\mathcal{G})^6 \\
&= 2^9 \cdot 4^{\nabla_1(\mathcal{G})} \cdot \nabla_1(\mathcal{G})^6,
\end{aligned}$$

as claimed. □

2.5 Domination and scattered sets

We now state the constant-factor approximation for Dominating Set proved by Dvořák [10]. The statement is slightly different from the results there, and we therefore explain how this exact statement can be derived from the work of Dvořák in the appendix.

Theorem 2.11. [$\star$] *There is a polynomial-time algorithm that given a graph G and an integer k, either finds a dominating set of size at most $2^{20}\nabla_1(G)^{12}k$ or a 2-scattered set of size at least $k+1$ in G.*

We remark that the proof of Theorem 2.11 does not assume that the graph belongs to some class of bounded expansion. If this is the case, then algorithm can be implemented with slightly better approximation ration and in linear time. However, in the nowhere dense case it will be important for us that we can apply Theorem 2.11 without this assumption, and in particular that the running time does not depend exponentially on the grads of G.

We need the following strengthened version of Dvořák's algorithm that approximates domination of only some subset of vertices.

Lemma 2.12. *There is a polynomial-time algorithm that, given a graph G, a vertex subset $Z \subseteq V(G)$ and an integer k, finds either*

- *a Z-dominator in G of size at most $2^{33}\nabla_1(G)^{12} \cdot k$, or*
- *a subset of Z of size at least $k+1$ that is 2-scattered in G.*

Proof. Obtain G' from G by adding first an isolated vertex v' and then a vertex v universal to $V(G') \setminus Z$ and adjacent to v'. Apply Theorem 2.11 to graph G' with parameter $k+1$.

Suppose first that the algorithm outputs a dominating set D in G'. By Lemma 2.3 and Theorem 2.11, this dominating set has size at most

$$2^{20}\nabla_1(G')^{12} \cdot (k+1) \leqslant 2^{20}(\nabla_1(G)+1)^{12} \cdot (k+1) \leqslant 2^{33}\nabla_1(G)^{12} \cdot k.$$

Observe that $D \cap V(G)$ is a Z-dominator in G; neither v nor v' can possibly dominate any vertex of Z.

Suppose now that the algorithm provided a 2-scattered set S in G' of size at least $k+2$. Observe that the graph $G' - Z$ has diameter 2 since v is a universal vertex for this graph. Hence any 2-scattered set in G' contains at most one vertex from $V(G) \setminus Z$. Therefore, S can contain at most one vertex outside of Z in G', hence $|S \cap Z| \geqslant k+1$ and $S \cap Z$ is the sought 2-scattered subset of Z. □

2.6 Nowhere dense graph classes

In this section we introduce auxiliary definitions and facts about nowhere dense graph classes. These results will be essential for the reasoning in Section 4, where we obtain an almost linear kernel for Dominating Set on any fixed nowhere dense class. However, no result of this section is used in Section 3 that treats of graph classes of bounded expansion, hence the reader only interested in bounded expansion graphs can omit this part.

We first introduce the definition of a nowhere dense graph class; recall that $\omega(G)$ denotes the size of the largest clique in G and $\omega(\mathcal{G}) = \sup_{G \in \mathcal{G}} \omega(G)$.

Definition 2.13 (Nowhere dense)**.** A graph class $\mathcal{G}$ then is *nowhere dense* if there exists a function $f_\omega \colon \mathbb{N} \to \mathbb{N}$ such that for all r we have that $\omega(\mathcal{G} \triangledown r) \leqslant f_\omega(r)$.

This definition follows closely the definition of bounded expansion. Since cliques have non-constant density, we have that every class of bounded expansion is also nowhere dense; however, the converse is not true [26].

We shall mostly rely on the following alternative characterization of nowhere dense graph classes, which follows easily from the following results of [25]: Theorem 4.1, points (ii) and (x), and Corollary 4.3.

Proposition 2.14 ([25])**.** *Let $\mathcal{G}$ be a nowhere dense graph class. Then:*

1. *There is a function $f_\nabla(r,\varepsilon)$ such that $\nabla_0(G') \leqslant f_\nabla(r,\varepsilon) \cdot |G'|^\varepsilon$ for every integer $r \geqslant 0$, $G' \in \mathcal{G} \triangledown r$, and real $\varepsilon > 0$. In particular, $\nabla_r(G) \leqslant f_\nabla(r,\varepsilon) \cdot |G|^\varepsilon$ for every integer $r \geqslant 0$, $G \in \mathcal{G}$, and real $\varepsilon > 0$.*
2. *There is a function $f_{\mathrm{wcol}}(r,\varepsilon)$ such that $\mathrm{wcol}_r(G) \leqslant f_{\mathrm{wcol}}(r,\varepsilon) \cdot |G|^\varepsilon$ for every integer $r \geqslant 0$, $G \in \mathcal{G}$, and real $\varepsilon > 0$.*

As shown in [25], conditions (1) and (2) are in fact equivalent to $\mathcal{G}$ being nowhere dense, provided that $\mathcal{G}$ is closed under taking subgraphs.

We remark that in the other literature on the topic, it is customary to use an alternative variant of this statement: for instance, there exists a constant $N^{\mathrm{wcol}}_{r,\varepsilon}$ such that $\mathrm{wcol}_r(G) \leqslant |G|^\varepsilon$ for any integer r, real ε and graph $G \in \mathcal{G}$ such that $|G| \geqslant N^{\mathrm{wcol}}_{r,\varepsilon}$; see e.g. [21, Lemma 5.3]. Whereas this formulation can be easily seen to be equivalent to ours, we find it more cumbersome to use in the proofs.

It turns out that the *clique density*, i.e. the number of complete subgraphs in a graph divided by the size of the graph, is an important measure that that determines the structure of nowhere dense graphs. Recall that $\#\omega(G)$ denotes the total number of cliques in G.

Lemma 2.15 (Clique density of nowhere dense graph)**.** *Let $\mathcal{G}$ be a nowhere dense class of graphs. Then there exists a function $f_{\#\omega}(r,\varepsilon)$ such that for any $G \in \mathcal{G}$, integer $r \geqslant 0$ and real $\varepsilon > 0$, we have that $\#\omega(G \triangledown r) \leqslant f_{\#\omega}(r,\varepsilon) \cdot |G|^{1+\varepsilon}$.*

Proof. Take any $H \in G \triangledown r$; of course, $|H| \leqslant |G|$. Since $G \in \mathcal{G}$, we have that $\omega(H) \leqslant f_\omega(r)$. By Proposition 2.14, point (2) applied to $r = 1$ and $\varepsilon' = \varepsilon/(f_\omega(r)-1)$, there exists an ordering $\sigma \in \Pi(H)$ such that for each $v \in V(H)$ we have that $B_1^{H,\sigma}(v) = \{u :\ u <_\sigma v \wedge uv \in E(H)\}$ has size at most $f_{\mathrm{wcol}}(1,\varepsilon') \cdot |H|^{\varepsilon'} \leqslant f_{\mathrm{wcol}}(1,\varepsilon') \cdot |G|^{\varepsilon'}$. For each clique $Q \subseteq V(H)$, let v_Q be the last vertex of Q in σ. Then we have that $Q \subseteq B_1^{H,\sigma}(v_Q)$. Therefore, for each $v \in V(H)$ we have that the number of cliques $Q \subseteq V(H)$ with $v = v_Q$ is at most

$$\sum_{d=0}^{f_\omega(r)-1} |B_1^{H,\sigma}(v)|^d \leqslant f_\omega(r) \cdot |B_1^{H,\sigma}(v)|^{f_\omega(r)-1} \leqslant f_\omega(r) \cdot f_{\mathrm{wcol}}(1,\varepsilon')^{f_\omega(r)-1} \cdot |G|^{\varepsilon}.$$

The claim follows by summing through all the vertices of H and using the fact that $|H| \leqslant |G|$. $\square$

We now use the following result from [19, Lemma 6.6] that relates the structure of bipartite graphs to the edge- and clique-density of its respective graph class.

Proposition 2.16. *Let $G = (X, Y, E)$ be a bipartite graph, and let $\mathcal{G}_1$ be the class of 1-shallow minors of G that have at most $|X|$ vertices. Let further $h = \sup_{H \in \mathcal{G}_1}(\#\omega(H)/|H|)$. Then there are at most*

1. *$2\nabla_0(\mathcal{G}_1) \cdot |X|$ vertices in Y with degree larger than $\omega(\mathcal{G}_1)$;*
2. *$(h + 2\nabla_0(\mathcal{G}_1)) \cdot |X|$ subsets $A \subseteq X$ such that $A = N(u)$ for some $u \in Y$.*

With these tools at hand, we can prove the following important lemma that serves the role of Proposition 2.5 in the nowhere dense case.

Lemma 2.17 (Twin classes)**.** *Let $\mathcal{G}$ be nowhere dense graph class. Then there exists a function $f_{nei}(\cdot)$ such that for any graph $G \in \mathcal{G}$, any nonempty vertex subset $X \subseteq V(G)$ and any $\varepsilon > 0$, the following holds:*

$$|\{A \subseteq X \colon \exists_{v \in V \setminus X}\ A = N_X(v)\}| \leqslant f_{nei}(\varepsilon) \cdot |X|^{1+\varepsilon}.$$

Proof. We would like to use the second bound of Proposition 2.16. Fix $\varepsilon > 0$, a graph $G \in \mathcal{G}$ and a nonempty vertex set $X \subseteq G$. Let G_0 be the bipartite graph $(X, V(G) \setminus X, E(G) \cap (X \times (V(G) \setminus X)))$. To obtain the sought bound, we need bounds on the quantities $h := \sup_{H \in \mathcal{G}_1}(\#\omega(H)/|H|)$ and $\nabla_0(\mathcal{G}_1)$, where $\mathcal{G}_1$ is defined for G_0 as in Proposition 2.16.

Since G_0 is a subgraph of G, we have that $\mathcal{G}_1 \subseteq G \triangledown 1$. Hence, from Lemma 2.15 we obtain

$$h = \sup_{H \in \mathcal{G}_1} \frac{f_{\#\omega}(H)}{|H|} \leqslant \sup_{\substack{H \in G \triangledown 1 \\ |H| \leqslant |X|}} \frac{f_{\#\omega}(1,\varepsilon)|H|^{1+\varepsilon}}{|H|} \leqslant f_{\#\omega}(1,\varepsilon) \cdot |X|^{\varepsilon}. \tag{1}$$

The bound of the grad follows directly from Proposition 2.14, point (1):

$$\nabla_0(\mathcal{G}_1) = \sup_{\substack{H \in G \triangledown 1 \\ |H| \leqslant |X|}} \nabla_0(H) \leqslant f_\nabla(1,\varepsilon) \cdot |X|^{\varepsilon}. \tag{2}$$

By plugging (1) and (2) in upper bound of Proposition 2.16 (2), we obtain that

$$|\{A \subseteq X \colon \exists_{v \in V \setminus X}\ A = N_X(v)\}| \leqslant (h + 2\nabla_0(\mathcal{G}_1)) \cdot |X| \leqslant (f_{\#\omega}(1,\varepsilon) + 2f_{\nabla}(1,\varepsilon)) \cdot |X|^{1+\varepsilon}.$$

Hence we can set $f_{\mathrm{nei}}(\varepsilon) = f_{\#\omega}(1,\varepsilon) + 2f_{\nabla}(1,\varepsilon)$. □

Given Lemma 2.17, we can now use it to prove the analogue of the Charging Lemma, i.e., Lemma 2.10. The proof uses the same approach via weak 2-colorings as that of Lemma 2.10, and is contained in the appendix for completeness.

Lemma 2.18. [$\star$] *Let $\mathcal{G}$ be a nowhere dense graph class. Then there exists a function $f_{chrg}(\cdot)$ such that the following holds. For any $\varepsilon > 0$ and any bipartite graph $G = (X, Y, E) \in \mathcal{G}$ such that every vertex from Y has a nonempty neighborhood in X and no two vertices of Y have the same neighborhood in X, there exists a mapping $\phi \colon Y \to X$ with the following properties:*

- *$u\phi(u) \in E$ for each $u \in Y$;*
- *$|\phi^{-1}(v)| \leqslant f_{chrg}(\varepsilon) \cdot |G|^{\varepsilon}$ for each $v \in X$.*

Finally, we state the variant of Dvořák's algorithm suitable for nowhere dense graphs. The following lemma follows directly from plugging the bound of Proposition 2.14, point (1), into Lemma 2.12.

Lemma 2.19. *Let $\mathcal{G}$ be a nowhere dense class of graphs. Then there exists a function $f_{dv}(\cdot)$ and a polynomial-time algorithm that, given a graph $G \in \mathcal{G}$, a vertex subset $Z \subseteq V(G)$ and an integer k, finds either:*

- *a Z-dominator in G that has size at most $f_{dv}(\varepsilon) \cdot k \cdot |G|^{\varepsilon}$ for every $\varepsilon > 0$, or*
- *a subset of Z of size at least $k+1$ that is 2-scattered in G.*

3 A kernel for graphs of bounded expansion

In this section we give a linear kernel for Dominating Set on graphs of bounded expansion; that is, we prove Theorem 1.1. Let us fix a graph class $\mathcal{G}$ that has bounded expansion, and let (G, k) be the input instance of Dominating Set, where $G \in \mathcal{G}$. We assume that $\nabla_0(\mathcal{G}) \geqslant 1$, otherwise G is a forest and the Dominating Set problem can be solved in linear time.

We assume that $\mathcal{G}$ is fixed and thus also the values of $\nabla_r(\mathcal{G})$ for $0 \leqslant r \leqslant 4$. We discuss in Section 6 that the values of $\nabla_r(\mathcal{G})$ for $0 \leqslant r \leqslant 4$ need not be known to the algorithm, but it will significantly simplify the analysis.

As explained in Section 1, the first goal is to reduce the number of dominatees. More precisely, we find a subset of vertices Z of size linear in k, called a *domination core*, such that any Z-dominator is guaranteed to dominate the whole graph. In this manner, domination of vertices outside the domination core is not relevant to the problem, and they can only serve the role of dominators. Reducing their number is performed in the second step of the algorithm.

3.1 Reducing dominatees

We begin with introducing formally the notion of a domination core:

Definition 3.1 (Domination core)**.** Let G be a graph and Z be a subset of vertices. We say that Z is a *domination core* in G if every minimum-size Z-dominator in G is also a dominating set in G.

Clearly, the whole $V(G)$ is a domination core, but we look for a domination core that is small in terms of k. Note that if Z is a domination core, then $\mathbf{ds}(G) = \mathbf{ds}(G, Z)$. Let us remark that in this definition we do not require that every Z-dominator is a dominating set in G; there can exist Z-dominators that are not of minimum size and that do not dominate the whole graph.

The rest of this subsection is devoted to the proof of the following theorem.

Theorem 3.2. *There exists a function $f_{coresize}(\cdot)$ and a polynomial-time algorithm that, given an instance (G, k) where $G \in \mathcal{G}$, either correctly concludes that $\mathbf{ds}(G) > k$, or finds a domination core $Z \subseteq V(G)$ with $|Z| \leqslant f_{coresize}(\nabla_4(\mathcal{G})) \cdot k$.*

We fix G and k in the following to improve readability. For the proof of Theorem 3.2 we start with $Z = V(G)$ and gradually reduce $|Z|$ by removing one vertex at a time, while maintaining the invariant that Z is a domination core. To this end, we need to prove the following lemma, from which Theorem 3.2 follows trivially as explained:

Lemma 3.3. *There exists a function $f_{coresize}(\cdot)$ and a polynomial-time algorithm that, given a domination core $Z \subseteq V(G)$ with $|Z| > f_{coresize}(\nabla_4(\mathcal{G})) \cdot k$, either correctly concludes that $\mathbf{ds}(G) > k$, or finds a vertex $z \in Z$ such that $Z \setminus \{z\}$ is still a domination core.*

Thus, from now on we focus on proving Lemma 3.3.

3.1.1 Iterative extraction of Z-dominators

The first phase of the algorithm is to build a structural decomposition of the graph G. More precisely, we try to "pull out" a small set X of vertices that dominates Z, so that after removing them, Z contains a large subset S, which is 2-scattered in the remaining graph. Given such a structure, we can argue that any optimal Z-dominator should take vertices from X (which dominate many vertices of S) rather than from $V(G) \setminus Z$ (which can dominate only at most one vertex from S). Since S will be large compared to X, some vertices of S will be indistinguishable from the point of view of domination from X, and these will be precisely the vertices that can be removed from the domination core. The identification of the irrelevant dominatee will be the goal of the second phase of the algorithm, whereas the goal of this phase is to construct the pair (X, S).

Given Z, we first apply the algorithm of Lemma 2.12 to G, Z, and the parameter k. Thus, we either find a Z-dominator X_1 such that $|X_1| \leqslant 2^{33}\nabla_1(G)^{12} \cdot k \leqslant 2^{33}\nabla_1(\mathcal{G})^{12} \cdot k$, or we find a subset $S \subseteq Z$ of size at least $k+1$ that is 2-scattered in G. In the latter case, since S is an obstruction to a dominating set of size at most k, we may terminate the algorithm and provide a negative answer. Hence, from now on we assume that X_1 has been successfully constructed.

Now, we inductively construct sets $X_2, X_3, X_4, \ldots$ such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$. We shall maintain the invariant

$$|X_i| \leqslant f_i(\nabla_4(\mathcal{G})) \cdot k,$$

where $(f_i)_{i=1,2,\ldots}$ is a sequence of upper bounds that increases quite rapidly with i. More precisely, we inductively define the following functions:

$$\begin{aligned} g(x) &= 2^{105} 2^{20x} x^{53} + 1, \\ f_i(x) &= (1 + (2^{33} x^{12} \cdot g(x)))^{i-1} \cdot 2^{33} x^{12}. \end{aligned}$$

Observe that we indeed have that $|X_1| \leqslant f_1(\nabla_1(\mathcal{G})) \cdot k \leqslant f_1(\nabla_4(\mathcal{G})) \cdot k$, so the invariant is satisfied in the first step.

We now explain how X_{i+1} is computed based on X_i:

1. First, apply the algorithm of Lemma 2.12 to $G - X_i$, $Z \setminus X_i$, and the parameter $g(\nabla_4(\mathcal{G})) \cdot |X_i|$.

2. Suppose the algorithm has found a set $S \subseteq Z \setminus X_i$, which is 2-scattered in $G - X_i$, and has cardinality greater than $g(\nabla_4(\mathcal{G})) \cdot |X_i|$. Then we let $X = X_i$, terminate the computation of the sets X_i and proceed to the second phase with the pair (X, S).

3. Otherwise, the algorithm has found a $(Z \setminus X_i)$-dominator D_{i+1} in $G - X_i$ such that

$$|D_{i+1}| \leqslant \left(2^{33}\nabla_1(G)^{12} \cdot g(\nabla_4(\mathcal{G}))\right) \cdot |X_i| \leqslant \left(2^{33}\nabla_1(\mathcal{G})^{12} \cdot g(\nabla_4(\mathcal{G}))\right) \cdot |X_i|$$

   We let $X_{i+1} = X_i \cup D_{i+1}$ and proceed to the next i. Observe that

$$\begin{aligned} |X_{i+1}| &\leqslant \left(1 + 2^{33}\nabla_1(\mathcal{G})^{12} \cdot g(\nabla_4(\mathcal{G}))\right) \cdot |X_i| \\ &\leqslant \left(1 + 2^{33}\nabla_4(\mathcal{G})^{12} \cdot g(\nabla_4(\mathcal{G}))\right) \cdot f_i(\nabla_4(\mathcal{G})) \cdot k \\ &= f_{i+1}(\nabla_4(\mathcal{G})) \cdot k. \end{aligned}$$

   Hence, the invariant that $|X_i| \leqslant f_i(\nabla_4(\mathcal{G})) \cdot k$ is maintained in the next iteration.

In this manner, the algorithm consecutively extracts Z-dominators $D_2, D_3, D_4, \ldots$ constructing sets $X_2, X_3, X_4, \ldots$ up to the point when case (2) is encountered. Then the computation is terminated and the sought pair (X, S) is constructed. We now claim that case (2) always happens within a constant number of iterations.

Lemma 3.4. *Assuming that $|Z| > 2f_{2\nabla_0(\mathcal{G})}(\nabla_4(\mathcal{G})) \cdot k$, the construction terminates with some pair (X, S) before iteration $2\nabla_0(\mathcal{G})$, that is, before constructing $X_{2\nabla_0(\mathcal{G})}$.*

Proof. Suppose for a contradiction that the algorithm actually performed $2\nabla_0(\mathcal{G}) - 1$ iterations and hence it constructed $Q = X_{2\nabla_0(\mathcal{G})}$. Then $|Q| \leqslant bk$ where $b = f_{2\nabla_0(\mathcal{G})}(\nabla_4(\mathcal{G}))$.

Since $|Z| > 2bk$, we infer that $|Z \setminus Q| > bk \geqslant |Q|$. Take any $z \in Z \setminus Q$, and observe that by the construction of $X_1, X_2, \ldots, X_{2\nabla_0(\mathcal{G})}$, it must hold that z has one neighbor in each of the sets $X_1, D_2, D_3 \ldots, D_{2\nabla_0(\mathcal{G})}$ constructed along the way: each of these sets is a Z-dominator. Thus, z must have at least $2\nabla_0(\mathcal{G})$ neighbors in Q. Construct a set P by taking Q together with $|Q| + 1$ arbitrarily chosen elements of $Z \setminus Q$. Observe that the density of $G[P]$ is at least $\frac{2\nabla_0(\mathcal{G}) \cdot (|Q|+1)}{2|Q|+1} > \nabla_0(\mathcal{G})$, which is a contradiction. □

Therefore, unless the size of Z is bounded by $2f_{2\nabla_0(\mathcal{G})}(\nabla_4(\mathcal{G})) \cdot k$, the construction terminates within $2\nabla_0(\mathcal{G}) - 1$ iterations with a pair (X, S). By the construction of X and S, we have the following properties:

- $|X| \leqslant 2 \cdot f_{2\nabla_0(\mathcal{G})-1}(\nabla_4(\mathcal{G})) \cdot k$;
- X is a Z-dominator in G;
- $|S| > (2^{105} 2^{20\nabla_4(\mathcal{G})} \nabla_4(\mathcal{G})^{53} + 1) \cdot |X|$;
- $S \subseteq Z \setminus X$ and S is 2-scattered in $G - X$.

With sets X and S computed we proceed to the second phase, that is, finding an irrelevant dominatee that can be removed from Z.

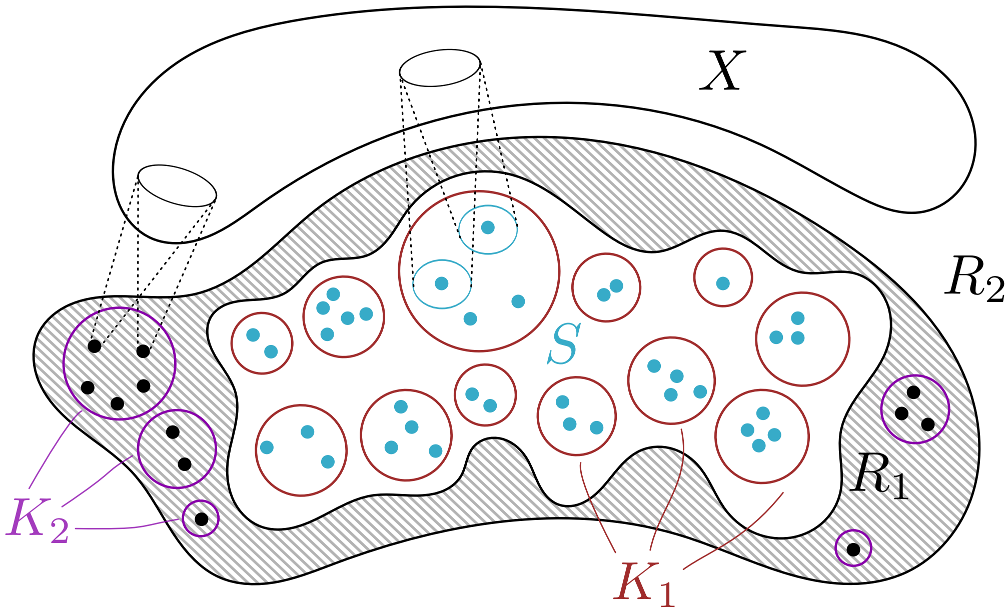

Figure 2: Overview over important vertex sets.

3.1.2 Finding an irrelevant dominatee

Given G and the constructed sets X, Z and S, we denote by $R = V(G) \setminus X$ the vertices outside X. Using this notation, S is 2-scattered in the graph $G[R]$. Recall that for any vertex $u \in R$, by the X*-neighborhood* of u, denoted $N_X(u)$, we mean $N(u) \cap X$.

We construct an auxiliary graph $G' \in G \,\triangledown\, 1$ as follows: for every vertex $s \in S$, we contract every vertex of the set $N(s) \setminus X$ into s. Since the vertices of S are 2-scattered in $G - X$, the sets $N(s) \setminus X$ are pairwise disjoint for different $s \in S$ and this operation creates a 1-shallow minor of G. The vertex of G' onto which the set $N(s) \setminus X$ is contracted to is renamed as s. We denote by $N'(\cdot)$ and $N'[\cdot]$, respectively, open and closed neighborhoods of vertices in G'. Again, the X-neighborhoods in G' are denoted $N'_X(u) = N'(u) \cap X$, for $u \in V(G')$.

Note that for a vertex $s \in S$, we have that $N_X(s) \subseteq N'_X(s)$. Moreover, both these sets are nonempty, since X is a Z-dominator in G and $S \subseteq Z$. We first prove that most vertices of S actually have few neighbors in X in the graph G'.

Lemma 3.5. *There are at most $|X|$ vertices $s \in S$ for which $|N'_X(s)| > 2\nabla_1(\mathcal{G})$.*

Proof. Assume that there are more than $|X|$ such vertices, and let S' be a set of $|X| + 1$ such vertices. Consider the induced subgraph $G'[S' \cup X]$. The density of this subgraph is at least

$$\frac{|S'| \cdot 2\nabla_1(\mathcal{G})}{|S'| + |X|} = \frac{|S'| \cdot 2\nabla_1(\mathcal{G})}{2|S'| - 1} > \nabla_1(\mathcal{G}),$$

which is impossible since G', and therefore also $G'[S' \cup X]$, are 1-shallow minors of $G \in \mathcal{G}$. ☐

We now remove from S all the vertices s with $|N'_X(s)| > 2\nabla_1(\mathcal{G})$; Lemma 3.5 ensures that there is at most $|X|$ of them, and hence the resulting set has size at least $|S| - |X| > 2^{105} 2^{20\nabla_4(\mathcal{G})} \nabla_4(\mathcal{G})^{53} \cdot |X|$. For ease of readability, we abuse notation and call the remaining set S. We also reconstruct the graph G' according to the new definition of S. Hence, from now on we assume that $|S| > 2^{105} 2^{20\nabla_4(\mathcal{G})} \nabla_4(\mathcal{G})^{53} \cdot |X|$ and that no vertex of S has more than $2\nabla_1(\mathcal{G})$ neighbors in X in G'.

Let $R_1 = R \cap N[S]$ be those vertices of R that can possibly dominate a vertex in S, and let $R_2 = R \setminus R_1$ be all the other vertices in R. We now partition the vertices of $G' - X$ into classes according to their neighborhoods in X. Note that by the construction of G', we have that $V(G' - X) = S \cup R_2$. We define the equivalence relation $\simeq_X$ over $S \cup R_2$ as follows:

$$u \simeq_X v \Leftrightarrow N'_X(u) = N'_X(v).$$

In the following, we consider the quotients (sets of classes of abstraction) $K_1 = S/{\simeq_X}$ and $K_2 = R_2/{\simeq_X}$. We will also use $K = K_1 \cup K_2$. Note that since vertices of R_2 are untouched during the construction of G', we have that K_2 is simply the partitioning of vertices of R_2 with respect to their X-neighborhoods in G. Each $\kappa \in K$ will simply be called a *class*. For a class $\kappa \in K$, by $N'_X(\kappa)$ we denote the common X-neighborhood of vertices of κ in G'.

Observe that each class $\kappa \in K_1$ consists of vertices from $S \subseteq Z$, which, since X is a Z-dominator, have to have neighbors in X in graph G. Hence, $N'_X(\kappa)$ is nonempty for each $\kappa \in K_1$. However, in K_2 there may be a class $\kappa_\emptyset$ whose vertices do not have neighbors in X; i.e., $N'_X(\kappa_\emptyset) = \emptyset$. Note that the vertices of this class, provided it exists, cannot be contained in Z.

For a class $\kappa \in K_1$ we define $U_\kappa = N[\kappa] \cap R$. That is, U_κ comprises all vertices of $R \subseteq V(G)$ that have been contracted onto the vertices of κ during the construction of G'. Since S is 2-scattered in $G[R]$, the sets U_κ for $\kappa \in K_1$ are pairwise disjoint. Moreover, $(U_\kappa)_{\kappa \in K_1}$ forms a partition of R_1.

Intuitively, our goal now is to identify a large class $\kappa \in K_1$ that cannot be dominated by a small set of vertices in R. We then argue that such a class contains a vertex that is irrelevant: it can be removed from Z without breaking the invariant that Z is a domination core.

First, we define an auxiliary graph that captures the interaction between the classes in K.

Definition 3.6. The *class graph* H is a graph with vertex set K that contains an edge between $\kappa, \kappa' \in K$ if and only if there exists $u \in \kappa$ and $u' \in \kappa'$ such that $uu' \in E(G')$.

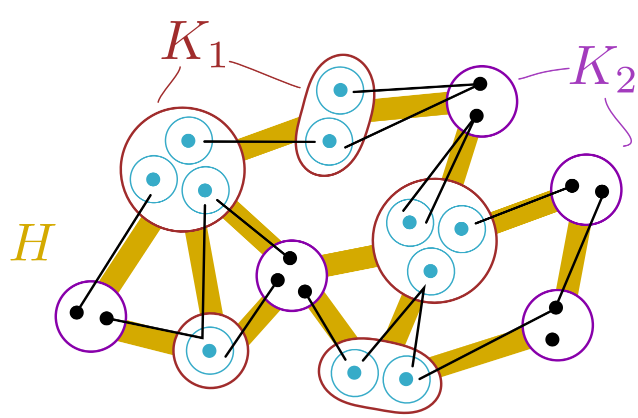

Figure 3: The class graph H with vertex set $K_1 \cup K_2 = K$.

The crucial observation is that the class graph actually cannot be too large and complicated: it has $O(|X|)$ vertices, and has bounded expansion. We now prove these facts formally.

Lemma 3.7 (Size of the class graph)**.** *The following holds:*

- $|K_1| \leqslant 2 \cdot 4^{\nabla_4(\mathcal{G})} \cdot |X|$, *and*
- $|K_2| \leqslant 2 \cdot 4^{\nabla_1(\mathcal{G})} \cdot |X|$.

Consequently, $|V(H)| = |K| \leqslant 4 \cdot 4^{\nabla_4(\mathcal{G})} \cdot |X|$.

Proof. Recall from Proposition 2.5 that for $G \in \mathcal{G}$ and $X \subseteq V(G)$, with $R = V(G) \setminus X$, we have that $|\{N_X(v) \colon v \in R\}| \leqslant 2 \cdot 4^{\nabla_1(\mathcal{G})} \cdot |X|$. The upper bound on $|K_2|$ (second item) follows from the above equation applied to the graph $G - R_1$ and X. In order to obtain an upper bound on $|K_1|$, we again apply the above equation, but now to the graph $G'[S \cup X]$ and X. We thus infer that the number of possible X-neighborhoods among the vertices in S, and hence the number of classes in K_1, is at most $2 \cdot 4^{\nabla_1(G')} \cdot |X|$. Since G' is a 1-shallow minor of $G \in \mathcal{G}$, by Proposition 2.4 it follows that $\nabla_1(G') \leqslant \nabla_4(\mathcal{G})$. Hence $|K_1| \leqslant 2 \cdot 4^{\nabla_4(\mathcal{G})} \cdot |X|$. □

Our intuition tells us that the grad of H should be related to the grad of G. Since H is not necessarily a shallow minor of G (the members of a class are not necessarily connected in G' and thus can therefore not necessarily be contracted), the proof of this fact needs some additional work. We remark that the only fact we use later is that H has a constant density; however, we find the observation that H has bounded expansion quite insightful.

Lemma 3.8 (Grad of the class graph)**.** *There exists a function $h(\cdot,\cdot)$ such that for every $r \geqslant 0$ it holds that $\nabla_r(H) \leqslant h(r, \nabla_{9r+4}(\mathcal{G}))$. In particular, $\nabla_0(H) \leqslant 2^{100} 2^{16\nabla_4(\mathcal{G})} \nabla_4(\mathcal{G})^{52}$.*

Before we proceed to the proof of Lemma 3.8, we briefly discuss the intuition behind it, which consists of two steps: We first blow up the graph using the lexicographic product with a constant-sized clique. The resulting grad is bounded, as was observed in Lemma 2.7. Then, we apply Lemma 2.10 in order to find a smart way of contracting vertices in such a manner that every class κ ends up being contracted onto a different copy of a vertex from X. The class graph H will then appear as a subgraph of this construction, and therefore we can bound its grad as a function of G's grad. During the construction we need to be careful about the class $\kappa_\emptyset$, which needs to be treated separately, since it cannot be contracted onto X at all.

Proof of Lemma 3.8. In the following, we assume that $\kappa_\emptyset$ exists. In case this class does not exist, the proof follows the same line of reasoning and is, in fact, simpler, as we do not need to perform the final step of considering $\kappa_\emptyset$ separately.

Let L_1 be a set constructed by selecting an arbitrary vertex s_κ from each $\kappa \in K_1$. Analogously construct a set L_2 for the set of classes $K_2 \setminus \{\kappa_\emptyset\}$. We define two bipartite graphs $G_1' = (L_1, X, E(G') \cap (L_1 \times X))$ and $G_2' = (L_2, X, E(G') \cap (L_2 \times X))$; observe that both of them are subgraphs of G'. Moreover, by the definition of the classes from K_1 and K_2 we infer that these graphs satisfy the assumptions of Lemma 2.10: X-neighborhoods of vertices in L_t are distinct and non-empty, for $t = 1, 2$.

Hence, we can apply Lemma 2.10 to the graphs G_1' and G_2', thus obtaining assignments $\phi_1 \colon L_1 \to X$ and $\phi_2 \colon L_2 \to X$ such that the pre-images of every vertex of X under ϕ_1 and ϕ_2 have sizes at most $2^9 4^{\nabla_1(G')} \nabla_1(G')^6$. As the vertices of L_1 and L_2 correspond bijectively to classes of K_1 and $K_2 \setminus \{\kappa_\emptyset\}$ respectively, we consider ϕ_1 and ϕ_2 also as assignments from K_1 to X and from $K_2 \setminus \{\kappa_\emptyset\}$ to X, respectively.

Let us combine ϕ_1, ϕ_2 into $\phi \colon K_1 \cup K_2 \setminus \{\kappa_\emptyset\} \to X$. Hence, the number of times a single vertex $v \in X$ can be chosen is at most doubled: for every $v \in X$, it holds that

$$|\phi^{-1}(v)| \leqslant 2^{10} 4^{\nabla_1(G')} \nabla_1(G')^6 \leqslant 2^{10} 4^{\nabla_4(\mathcal{G})} \nabla_4(\mathcal{G})^6,$$

where the last inequality follows from Proposition 2.4. Let $\tau = 2^{10} 4^{\nabla_4(\mathcal{G})} \nabla_4(\mathcal{G})^6$. We now consider the lexicographic product $G'' = G' \bullet K_\tau$. Let us construct a 1-shallow minor $H' \in G'' \,\triangledown\, 1$ as follows: for every class $\kappa \in K \setminus \{\kappa_\emptyset\}$, contract all the copies of all the vertices of κ onto one of the copies of $\phi(\kappa) \in X$, so that every class $\kappa \in K \setminus \{\kappa_\emptyset\}$ is contracted onto a different vertex. Since every vertex of X is chosen at most τ times by ϕ, such a contraction is possible. Let $\overline{\phi}\colon K \setminus \{\kappa_\emptyset\} \to V(G'')$ be an injection that assigns classes of $K \setminus \{\kappa_\emptyset\}$ to the copies of vertices of X they are contracted onto. Then it is easy to see that $\overline{\phi}$ defines a subgraph embedding of $H - \{\kappa_\emptyset\}$ into H'. Consequently, $H - \{\kappa_\emptyset\}$ is a 1-shallow minor of G'', and hence we can bound the grads of $H - \{\kappa_\emptyset\}$ using Proposition 2.4 and Lemma 2.7:

$$\begin{aligned}\nabla_r(H - \{\kappa_\emptyset\}) &\leqslant \nabla_{3r+1}(G'') \leqslant 4(8\tau(3r+1+\tau)\nabla_{3r+1}(G') + 4\tau)^{(3r+2)^2} \\ &\leqslant 4(8\tau(3r+1+\tau)\nabla_{9r+4}(G) + 4\tau)^{(3r+2)^2} \\ &\leqslant 4(8\tau(3r+1+\tau)\nabla_{9r+4}(\mathcal{G}) + 4\tau)^{(3r+2)^2}.\end{aligned}$$

To obtain a bound on the grads of H, observe that H is a subgraph of the graph obtained from $H - \{\kappa_\emptyset\}$ by adding a universal vertex. From Lemma 2.3 we infer that $\nabla_r(H) \leqslant \nabla_r(H - \{\kappa_\emptyset\}) + 1$, and hence

$$\nabla_r(H - \{\kappa_\emptyset\}) \leqslant 4(8\tau(3r+1+\tau)\nabla_{9r+4}(\mathcal{G}) + 4\tau)^{(3r+2)^2} + 1 = h(r, \nabla_{9r+4}(\mathcal{G})),$$

since τ is a function of r and $\nabla_4(\mathcal{G}) \leqslant \nabla_{9r+4}(\mathcal{G})$.

Thus, we have proven the first part of the lemma. The second part, i.e., the explicit bound on the degeneracy of H, follows from taking $r = 0$ and substituting $\tau = 2^{10}4^{\nabla_4(\mathcal{G})}\nabla_4(\mathcal{G})^6$:

$$\begin{aligned}\nabla_0(H) &\leqslant 4(8\tau(1+\tau)\nabla_4(\mathcal{G}) + 4\tau)^4 + 1 \\ &\leqslant 4\nabla_4(\mathcal{G})^4 \cdot (8\tau^2 + 12\tau)^4 + 1 \\ &\leqslant 4\nabla_4(\mathcal{G})^4 \cdot (20\tau^2)^4 + 1 \\ &\leqslant 2^{20} \cdot \nabla_4(\mathcal{G})^4 \cdot \tau^8 \\ &= 2^{100} \cdot \nabla_4(\mathcal{G})^{52} \cdot 2^{16\nabla_4(\mathcal{G})}.\end{aligned}$$

□

As a corollary to Lemmas 3.7 and 3.8, we can now bound the number of edges in the class graph by a linear function of $|X|$.

Corollary 3.9. *The number of edges in H, $|E(H)|$, is at most $2^{102}\nabla_4(\mathcal{G})^{52}2^{18\nabla_4(\mathcal{G})}|X|$.*

Proof. Since $|E(H)| \leqslant \nabla_0(H) \cdot |V(H)|$, we apply the upper bounds proven in Lemmas 3.7 and 3.8 and obtain

$$\begin{aligned}|E(H)| &\leqslant 2^{100}\nabla_4(\mathcal{G})^{52}2^{16\nabla_4(\mathcal{G})} \cdot 4 \cdot 4^{\nabla_4(\mathcal{G})} \cdot |X| \\ &\leqslant 2^{102}\nabla_4(\mathcal{G})^{52}2^{18\nabla_4(\mathcal{G})} \cdot |X|.\end{aligned}$$

□

Our next goal is to find a class $\kappa \in K_1$ which is large compared to its degree in the class graph H. This class is the set of candidates among which an irrelevant vertex will be identified.

Lemma 3.10 (Large subclass)**.** *There exists a class $\kappa \in K_1$ and a subset $\lambda \subseteq \kappa$ with the property that every member has the same X-neighborhood in G and, furthermore, that*

$$|\lambda| > 2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1) + 1.$$

Proof. For $\kappa \in K_1$ let us define the following potential function:

$$\Phi(\kappa) = |\kappa| - 2^{2\nabla_1(\mathcal{G})}(2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1) + 1).$$

Summing over all $\kappa \in K_1$ we observe that

$$\begin{aligned}\sum_{\kappa \in K_1} \Phi(\kappa) &= \sum_{\kappa \in K_1} |\kappa| - 2^{2\nabla_1(\mathcal{G})} \sum_{\kappa \in K_1} (2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1) + 1) \\ &= |S| - 2^{2\nabla_1(\mathcal{G})} \left(2\nabla_1(\mathcal{G}) \sum_{\kappa \in K_1} \deg_H(\kappa) + (2\nabla_1(\mathcal{G}) + 1) \cdot |K_1| \right).\end{aligned}$$

Using the fact that $\sum_{\kappa \in K_1} \deg_H(\kappa) \leqslant \sum_{\kappa \in V(H)} \deg_H(\kappa) = 2|E(H)|$ and the upper bound from Corollary 3.9, we obtain

$$\sum_{\kappa \in K_1} \Phi(\kappa) \geqslant |S| - 2^{2\nabla_1(\mathcal{G})} \left(2^{104} \nabla_4(\mathcal{G})^{53} 2^{18\nabla_4(\mathcal{G})} \cdot |X| + (2\nabla_1(\mathcal{G}) + 1)|K_1| \right).$$

Now we apply the upper bound on $|K_1|$ in terms of $|X|$ from Lemma 3.7:

$$\begin{aligned}\sum_{\kappa \in K_1} \Phi(\kappa) &\geqslant |S| - 2^{2\nabla_1(\mathcal{G})} \left(2^{104} \nabla_4(\mathcal{G})^{53} 2^{18\nabla_4(\mathcal{G})} + 2 \cdot 4^{\nabla_4(\mathcal{G})} (2\nabla_1(\mathcal{G}) + 1) \right) \cdot |X| \\ &\geqslant |S| - 2^{105} \nabla_4(\mathcal{G})^{53} 2^{20\nabla_4(\mathcal{G})} \cdot |X|.\end{aligned}$$

However, recall that $|S| > 2^{105} \nabla_4(\mathcal{G})^{53} 2^{20\nabla_4(\mathcal{G})} \cdot |X|$ by the construction of S, so we conclude that $\sum_\kappa \Phi(\kappa) > 0$. Therefore, there exists at least one $\kappa \in K_1$ for which $\Phi(\kappa) > 0$. Equivalently,

$$|\kappa| > 2^{2\nabla_1(\mathcal{G})}(2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1) + 1).$$

Having found such a large class κ, we proceed as follows. Partition the members of κ according to their X-neighborhoods N_X in G; recall that the members of κ have the same X-neighborhood N'_X in G', but they potentially have different X-neighborhoods in G. However, since we explicitly excluded from S all the vertices that have more than $2\nabla_1(\mathcal{G})$ neighbors in X in graph G', we know that $|N'_X(\kappa)| \leqslant 2\nabla_1(\mathcal{G})$ (see Lemma 3.5 and the subsequent paragraph). Since $N_X(s) \subseteq N'_X(\kappa)$ for each $s \in \kappa$, the number of possible classes we partition κ into is trivially bounded by $2^{|N'_X(s)|} \leqslant 2^{2\nabla_1(\mathcal{G})}$. Thus, we arrive at the conclusion that one of the parts $\lambda \subseteq \kappa$ satisfies the size bound

$$|\lambda| > 2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1) + 1$$

as claimed. □

We can finally state the lemma that identifies the irrelevant vertex:

Lemma 3.11. *Let z be an arbitrary vertex of λ. Then $Z \setminus \{z\}$ is still a domination core.*

Proof. Denote $Z' = Z \setminus \{z\}$ and let $N_X(\lambda) \subseteq N'_X(\kappa)$ be equal to $N_X(s)$ for any $s \in \lambda$ (these X-neighborhoods are by definition equal). Let D be any minimum-size Z'-dominator in G; the goal is to prove that D is a dominating set in G. In the following we work in the graph G.

Assume first that D contains some vertex from $N_X(\lambda)$. Since every vertex of $N_X(\lambda)$ is by definition adjacent to every vertex of λ, we have that D also dominates z. Hence D is a Z-dominator, and it must be of minimum size since $\mathbf{ds}(G, Z) \geqslant \mathbf{ds}(G, Z')$. Since Z was a domination core, we have that D is a dominating set in G.

Now suppose that $D \cap N_X(\lambda) = \emptyset$. We prove that this situation is impossible, as it would contradict the assumption that D has minimum size. Recall that for $\kappa_1 \in K_1$, we denote by $U_{\kappa_1} = N[\kappa_1] \cap R$ the set of all the vertices of R that can potentially dominate a vertex of κ_1.

Since $\lambda \setminus \{z\} \subseteq S \setminus \{z\} \subseteq Z'$, all the vertices of $\lambda \setminus \{z\}$ must be dominated somehow by D. They are not dominated from X, since we assumed that $D \cap N_X(\lambda) = \emptyset$. Hence, each vertex of $\lambda \setminus \{z\}$ has to be dominated by a vertex belonging to U_κ. Since S is 2-scattered in $G - X$, each element of D can dominate at most one element of $\lambda \setminus \{z\}$. We infer that $|D \cap U_\kappa| \geqslant |\lambda \setminus \{z\}| > 2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1)$.

Now construct a vertex set D' from D by:

- removing from D all the vertices belonging to U_κ;
- adding all the vertices of $N'_X(\kappa_1)$ for each $\kappa_1 \in N_H[\kappa] \cap K_1$; and
- adding an arbitrary vertex of $N_X(\kappa_2)$ for each $\kappa_2 \in N_H[\kappa] \cap K_2$, provided that the set $N_X(\kappa_2)$ is nonempty.

Clearly, in the first step we have removed more than $2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1)$ vertices. Observe that in the second we have added at most $2\nabla_1(\mathcal{G}) \cdot (\deg_H(\kappa) + 1)$ vertices: there are $\deg_H(\kappa) + 1$ classes in $N_H[\kappa]$, each class $\kappa_1 \in N_H[\kappa] \cap K_1$ has $|N'_X(\kappa_1)| \leqslant 2\nabla_1(\mathcal{G})$ and for each class $\kappa_2 \in N_H[\kappa] \cap K_2$ we have added at most $1 \leqslant 2\nabla_1(\mathcal{G})$ vertex from $N_X(\kappa_2)$. Thus, we have that $|D'| < |D|$.

To arrive at a contradiction it remains to show that D' is a Z'-dominator. By removing the vertices of $D \cap U_\kappa$ from D we might have removed domination from some vertices of Z' that are contained either (a) in $N'_X(\kappa)$, or (b) in U_κ, or (c) in U_{κ_1} for some $\kappa_1 \in N_H(\kappa) \cap K_1$, or (d) in some $\kappa_2 \in N_H(\kappa) \cap K_2$. Let $u \in Z'$ be some vertex from which we removed domination when removing $D \cap U_\kappa$ from D. We now investigate the aforementioned four cases, and in each case we prove that u is dominated by D'. Note that since $u \in Z$ and X is a Z-dominator, then u has at least one neighbor in X.

Case (a): $u \in N'_X(\kappa)$.
However, we have included $N'_X(\kappa)$ in D', so in particular $u \in D'$.

Case (b): $u \in U_\kappa$.
Note that every neighbor of u in X belongs to $N'_X(\kappa)$, and we have included the whole $N'_X(\kappa)$ in D'. As we have argued, u indeed has at least one neighbor in X. Hence this neighbor belongs to D' and dominates u.

Case (c): $u \in U_{\kappa_1}$ for some $\kappa_1 \in N_H(\kappa) \cap K_1$.
The same argumentation as in Case (b): Every neighbor of u in X belongs to $N'_X(\kappa_1)$, and we have included the whole $N'_X(\kappa_1)$ in D'. Moreover, u indeed has at least one neighbor in X, and hence this neighbor belongs to D' and dominates u.

Case (d): $u \in \kappa_2$ for some $\kappa_2 \in N_H(\kappa) \cap K_2$.
By the definition of K_2, the X-neighborhood of u is exactly $N_X(\kappa_2)$. Moreover, since u has at least one neighbor in X, this X-neighborhood is non-empty and hence we have picked at least one vertex from it to D'. Hence, the picked vertex belongs to D' and dominates u.

Therefore, $|D'| < |D|$ and D' is a Z'-dominator, contradicting the minimality of D. □

We now conclude the proof of Lemma 3.3, which also concludes the proof of Theorem 3.2. We let $f_{\text{coresize}}(\nabla_4(\mathcal{G})) = 2f_{2\nabla_0(\mathcal{G})}(\nabla_4(\mathcal{G}))$, for the reasoning of Section 3.3 to apply to any set Z of size more than $f_{\text{coresize}}(\nabla_4(\mathcal{G}))$. The algorithm works as follows: First, we compute the two sets X and S as described in Section 3.3; this can clearly be done in polynomial time since the construction boils down to a constant number of applications of the algorithm of Lemma 2.12. Then we compute the class graph H and identify a class κ and a subset $\lambda \subseteq \kappa$ that satisfy the statement of Lemma 3.10. Again, the construction of H can be done in polynomial time, and finding κ and λ requires iterating through all the classes of K_1, and then partitioning the vertices of κ according to their X-neighborhoods in G. Finally, Lemma 3.11 ensures that any vertex of λ can be output by the algorithm as an irrelevant dominatee.

3.2 Reducing dominators

Having reduced the number of vertices whose domination is essential, we arrive at the situation where the vast majority of vertices serve only the role of dominators. Now, it is relatively easy to reduce their number in one step, thus obtaining the sought kernel. In other words, we can now proceed to the proof of the main result; for the reader's convenience, we recall its statement.

Theorem 1.1. *Let $\mathcal{G}$ be a graph class of bounded expansion. There exists a polynomial-time algorithm that given a graph $G \in \mathcal{G}$ and an integer k, either correctly concludes that $\mathbf{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size $\mathcal{O}(k)$ with the property that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$.*

Proof. The algorithm works as follows. First, we apply the algorithm of Theorem 3.2 to compute a small domination core in the graph. In case that algorithm gives a negative answer, we output that $\mathbf{ds}(G) > k$. Hence, from here on, we assume that we have correctly computed a domination core $Z \subseteq V(G)$ of size at most $2f_{2\nabla_0(\mathcal{G})}(\nabla_4(\mathcal{G})) \cdot k$.

Partition the vertices of $V(G) \setminus Z$ into classes with respect to their Z-neighborhoods. By Proposition 2.5, the number of these classes is at most

$$\left(4^{\nabla_1(\mathcal{G})} + 4\nabla_1(\mathcal{G})\right) \cdot |Z| \leqslant 2 \cdot 4^{\nabla_1(\mathcal{G})} \cdot |Z|.$$

Recall that since Z is a domination core, it follows that $\mathbf{ds}(G) = \mathbf{ds}(G, Z)$.

Construct a set Y by taking Z and adding an arbitrarily selected vertex v_κ from each nonempty class κ of the introduced partition. We claim that Y satisfies the condition from the statement. Observe that

$$|Y| \leqslant 2 \cdot 4^{\nabla_1(\mathcal{G})}|Z| + |Z| \leqslant \left(2 \cdot 4^{\nabla_1(\mathcal{G})} + 1\right) \cdot 2f_{2\nabla_0(\mathcal{G})}(\nabla_4(\mathcal{G})) \cdot k,$$

so indeed $|Y| = \mathcal{O}(k)$.

Suppose first that $\mathbf{ds}(G) \leqslant k$. Let D be a minimum-size dominating set in G, so $|D| = \mathbf{ds}(G) = \mathbf{ds}(G, Z)$. It follows that D is a minimum-size Z-dominator as well. We construct D' by replacing $\kappa \cap D$ with v_κ for each class κ that has a nonempty intersection with D. Clearly, $|D'| \leqslant |D| = \mathbf{ds}(G, Z)$ and $D' \subseteq Y$. Moreover, D' is still a Z-dominator in G; the representative vertex v_κ dominates exactly the same vertices in Z as the vertices from $D \cap \kappa$. Therefore, since $|D'| \leqslant \mathbf{ds}(G, Z)$, it must hold that D' is a minimum-size Z-dominator in G and $|D'| = \mathbf{ds}(G, Z)$. Since Z is a domination core, we infer that D' is a dominating set in G. As D' is a dominating set in G and $D' \subseteq Y$, it follows that D' is a dominating set in $G[Y]$. Hence $\mathbf{ds}(G[Y]) \leqslant |D'| = \mathbf{ds}(G, Z) = \mathbf{ds}(G) \leqslant k$.

In the reverse direction, suppose now that $\mathbf{ds}(G[Y]) \leqslant k$ and let D' be a minimum-size dominating set in $G[Y]$; it follows that $|D'| = \mathbf{ds}(G[Y])$. In particular, D' dominates $Z \subseteq Y$, and hence it is a Z-dominator in G. It follows that $\mathbf{ds}(G, Z) \leqslant |D'| = \mathbf{ds}(G[Y]) \leqslant k$. Since $\mathbf{ds}(G) = \mathbf{ds}(G, Z)$, we get that $\mathbf{ds}(G) \leqslant k$ and we are done. ☐

4 A kernel for nowhere dense graphs

In this section we generalize the approach presented in Section 3 to give an almost linear kernel for Dominating Set in nowhere dense graph classes. In other words, we prove Theorem 1.2. The proof will closely follow the reasoning in the bounded expansion case, and in the presentation we assume that the reader is already familiar with the proof of Section 3. We need, however, to modify the reasoning in a few places.

From now on, we assume that $\mathcal{G}$ is a fixed nowhere dense graph class. Without loss of generality we assume that $\mathcal{G}$ is closed under taking subgraphs, since otherwise we may consider the closure of $\mathcal{G}$ under this operation, which is also nowhere dense. We fix all the functions given by Proposition 2.14 and Lemmas 2.17, 2.18, 2.19 for the class $\mathcal{G}$. Observe that the class $\mathcal{G} \triangledown 1$ is also nowhere dense, hence we can apply these results also to this class. We therefore fix also the functions given by Proposition 2.14 and Lemmas 2.17, 2.18, 2.19 for $\mathcal{G} \triangledown 1$, and we shall denote them by $f^1_\triangledown(\cdot,\cdot)$, $f^1_{\mathrm{nei}}(\cdot)$, $f^1_{\mathrm{chrg}}(\cdot)$ etc. Moreover, since $\mathcal{G}$ is nowhere dense, there exist constants c and c' such that $K_{c,c} \notin \mathcal{G} \triangledown 0$ and $K_{c',c'} \notin \mathcal{G} \triangledown 1$; in the following we shall use these constants extensively.

We also fix the real value $\varepsilon > 0$ for which the algorithm is constructed. Recall that Theorem 1.2 asserts the existence of an algorithm for each fixed value of ε, and not an algorithm that gets ε on the input. Thus, the values of functions given by Proposition 2.14 and Lemmas 2.17, 2.18, 2.19 for classes $\mathcal{G}$ and $\mathcal{G} \triangledown 1$ applied to any fixed ε' depending on ε can be hard-coded in the algorithm, and do not need to be computed. If we would like to implement one algorithm that works for ε given on the input, then we would need to assume that class $\mathcal{G}$ is *effectively nowhere dense*, that is, that function $f(r)$ in Definition 2.13 is computable. Then we would be able to derive that all the functions introduced in Section 2.6 are computable as well.

Let (G,k) be the input instance of Dominating Set such that $G \in \mathcal{G}$. We denote $n = |V(G)|$.

4.1 Reducing dominatees

Exactly as in Section 3, we are going to reduce the number of vertices whose domination is essential in the graph to almost linear in terms of k. More formally, we are going to find domination core that has size bounded by $g(\varepsilon) \cdot k \cdot n^\varepsilon$, for some function $g(\cdot)$ and every $\varepsilon > 0$. In this proof we shall use the same definition of a domination core as in Section 3; that is, we prove the following result:

Theorem 4.1. *There exists a function $g(\cdot)$ and a polynomial-time algorithm that, given an instance (G,k) where $G \in \mathcal{G}$ and any $\varepsilon > 0$, either correctly concludes that $\mathbf{ds}(G) > k$, or finds a domination core $Z \subseteq V(G)$ with $|Z| \leqslant g(\varepsilon) \cdot k \cdot n^\varepsilon$.*

Again, the proof of Theorem 4.1 follows trivially from iterative application of the following lemma that enables us to identify a vertex that can be safely removed from the domination core.

Lemma 4.2. *There exists a function $g(\cdot)$ and a polynomial-time algorithm that, given any $\varepsilon > 0$, a vertex subset $Z \subseteq V(G)$ with $|Z| > g(\varepsilon) \cdot k \cdot n^\varepsilon$ and a promise that Z is a domination core, either correctly concludes that $\mathbf{ds}(G) > k$, or finds a vertex $z \in Z$ such that $Z \setminus \{z\}$ is still a domination core.*

From now on we focus on proving Lemma 4.2. We fix the constant $\varepsilon > 0$ given to the algorithm; without loss of generality we assume that $\varepsilon < 1/10$. That is, all the constants introduced in the sequel may depend on ε.

4.1.1 Iterative extraction of Z-dominators

We now present the analogue of the subroutine presented in Section 3.1.1 for the nowhere dense case. The argumentation will differ in some important details. Before we proceed to formal details, let us begin with an informal discussion about these differences.

As in the bounded expansion case, the goal is to find a pair of disjoint subsets X and S with the following properties: X is bounded linearly in terms of $k \cdot n^{\varepsilon/2}$, whereas S is 2-scattered in $G - X$ and is at least $C \cdot n^{\varepsilon'}$ times larger than X, for some $\varepsilon' > 0$ and a constant C chosen as large as we like. If we now generalize the reasoning of Section 3.1.1 directly to the nowhere dense case, then every consecutive Z-dominator X_i would be $f(\delta) \cdot n^{\delta}$ times larger than the previous one, for any $\delta > 0$. As $\nabla_0(G)$ is not bounded by a constant anymore, in a direct generalization we would have problems with proving the analogue of Lemma 3.4: the argument that the construction terminates after a constant number of iterations breaks per se. We therefore replace it with a different argument based on discovering a large biclique subgraph in case the procedure runs for too many iterations.

We now proceed to the formal argumentation. Let $\delta = \frac{\varepsilon}{4c} > 0$ and let us fix some constant C, to be decided later. First, we apply Lemma 2.19 to G, Z, and parameters k and δ. This algorithm either outputs a subset $S \subseteq Z$ such that $|S| > k$ and S is 2-scattered in G, or a Z-dominator X_1 such that $|X_1| \leqslant f_{\mathrm{dv}}(\delta) \cdot k \cdot n^{\delta}$. In case S is found, every vertex of G can dominate at most one vertex of S and thus we can conclude that $\mathbf{ds}(G, Z) > k$. As $\mathbf{ds}(G, Z) = \mathbf{ds}(G)$, we infer that $\mathbf{ds}(G) > k$ and we can terminate the algorithm and provide a negative answer. Hence, from now on we assume that the Z-dominator X_1 has been successfully constructed.

Now, we inductively construct Z-dominators $X_2, X_3, X_4, \ldots$ such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq X_4 \subseteq \ldots$. We maintain the invariant that

$$|X_i| \leqslant C_i \cdot k \cdot n^{(2i-1)\delta},$$

where constants C_i are defined as

$$C_i := (1 + f_{\mathrm{dv}}(\delta) \cdot C)^{i-1} \cdot f_{\mathrm{dv}}(\delta).$$

Observe that $|X_1| \leqslant f_{\mathrm{dv}}(\delta) \cdot k \cdot n^{\delta}$, which means that the invariant is satisfied at the first step. We now describe how X_{i+1} is constructed based on X_i for consecutive $i = 1, 2, 3, \ldots$.

1. First, apply the algorithm of Lemma 2.19 to graph $G - X_i$, set $Z \setminus X_i$, and parameter $C \cdot |X_i| \cdot n^{\delta}$.
2. Suppose the algorithm has found a set $S \subseteq Z \setminus X_i$ that is 2-scattered in $G \setminus X_i$ and has cardinality larger than $C \cdot |X_i| \cdot n^{\delta}$. We set $X = X_i$, terminate the construction of sets X_i and proceed to the second phase with the pair (X, S).
3. Otherwise, the algorithm has found a $(Z \setminus X_i)$-dominator D_{i+1} in $G \setminus X_i$ such that

$$\begin{aligned} |D_{i+1}| &\leqslant f_{\mathrm{dv}}(\delta) \cdot C \cdot |X_i| \cdot n^{\delta} \cdot n^{\delta} \\ &= f_{\mathrm{dv}}(\delta) \cdot C \cdot |X_i| \cdot n^{2\delta} \end{aligned}$$

   We set $X_{i+1} = X_i \cup D_{i+1}$ and proceed to the next i. Observe that

$$\begin{aligned} |X_{i+1}| = |X_i| + |D_{i+1}| &\leqslant (1 + f_{\mathrm{dv}}(\delta) \cdot C) \cdot |X_i| \cdot n^{2\delta} \\ &\leqslant (1 + f_{\mathrm{dv}}(\delta) \cdot C) \cdot C_i \cdot k \cdot n^{(2i-1)\delta} \cdot n^{2\delta} \\ &= C_{i+1} \cdot k \cdot n^{(2(i+1)-1)\delta}. \end{aligned}$$

   Hence, the invariant that $|X_i| \leqslant C_i \cdot k \cdot n^{(2i-1)\delta}$ is maintained in the next iteration.

We now present the analogue of Lemma 3.4: we prove that the construction terminates by outputting a pair (X, S) after a constant number of iterations.

Lemma 4.3. *Assuming that $|Z| > (c \cdot f_{nei}(\varepsilon/2) + 1) \cdot C_c \cdot k \cdot n^{\varepsilon}$, the construction terminates by outputting some pair (X, S) after at most $c - 1$ iterations, i.e., before constructing X_c.*

Proof. For the sake of contradiction, suppose that the procedure actually performed $c-1$ iterations, successfully constructing disjoint Z-dominators $X_1, D_2, D_3, \ldots, D_c$, where $X_i = X_1 \cup D_2 \cup D_3 \cup \ldots \cup D_i$ for $i = 1, 2, \ldots, c$. Let $Q := X_c$ and observe that

$$\begin{aligned} |Z \setminus Q| \geqslant |Z| - |Q| &> (c \cdot f_{\mathrm{nei}}(\varepsilon/2) + 1) \cdot C_c \cdot k \cdot n^{\varepsilon} - C_c \cdot k \cdot n^{(2c-1)\delta} \\ &\geqslant (c \cdot f_{\mathrm{nei}}(\varepsilon/2) + 1) \cdot C_c \cdot k \cdot n^{\varepsilon} - C_c \cdot k \cdot n^{\varepsilon/2} \\ &\geqslant c \cdot f_{\mathrm{nei}}(\varepsilon/2) \cdot C_c \cdot k \cdot n^{\varepsilon} \geqslant c \cdot f_{\mathrm{nei}}(\varepsilon/2) \cdot |Q| \cdot n^{\varepsilon/2}. \end{aligned}$$

Now, partition vertices of $Z \setminus Q$ into classes with respect to their neighborhoods in Q. By Lemma 2.17, we infer that the number of these classes is at most $f_{\mathrm{nei}}(\varepsilon/2) \cdot |Q| \cdot n^{\varepsilon/2}$. Since $|Z \setminus Q| > c \cdot f_{\mathrm{nei}}(\varepsilon/2) \cdot |Q| \cdot n^{\varepsilon/2}$, we infer that one of these classes κ satisfies $|\kappa| \geqslant c$. However, each member of κ has neighbors in each of the Z-dominators $X_1, D_2, D_3, \ldots, D_c$, and hence the common Q-neighborhood of vertices of κ is of size at least c. Thus we see that the induced subgraph $G[\kappa \cup N_Q(\kappa)]$ contains a $K_{c,c}$ as a subgraph, a contradiction. □

Hence, provided that the cardinality of Z satisfies the lower bound stated in Lemma 4.3, the construction will terminate after at most $c - 1$ iterations, thus constructing sets X and S with the following properties:

- $|X| \leqslant C_{c-1} \cdot k \cdot n^{\varepsilon/2}$;
- X is a Z-dominator in G;
- $|S| > C \cdot |X| \cdot n^{\delta}$;
- $S \subseteq Z \setminus X$ and S is 2-scattered in $G - X$.

With sets X and S we proceed to the second phase, that is, finding an irrelevant dominatee.

4.1.2 Finding an irrelevant dominatee

We again denote $R := V(G) \setminus X$ and we define $R_1 := \bigcup_{s \in S} N[s] \cap R$ and $R_2 := R \setminus R_1$. Then $S \subseteq Z \cap R$ and S is 2-scattered in $G[R]$. Again, graph G' is a 1-shallow minor of G constructed by contracting every vertex of $N(s) \cap R$ onto s, for each $s \in S$; the resulting vertex is identified with s. We adopt the same notation for neighborhoods as in Section 3: $N'(\cdot)$ and $N'_X(\cdot)$ denote neighborhoods and X-neighborhoods in graph G', respectively.

As before, we claim that only few vertices of S can have large X-neighborhoods in G'.

Lemma 4.4. *The number of vertices $s \in S$ for which $|N'_X(s)| \geqslant c'$ holds is at most $c' \cdot f^1_{nei}(\delta) \cdot |X| \cdot n^{\delta}$.*

Proof. Let $S' = \{s \,:\, s \in S \wedge |N'_X(s)| \geqslant c'\}$, and for the sake of contradiction suppose $|S'| > c' \cdot f^1_{\mathrm{nei}}(\delta) \cdot |X| \cdot n^{\delta}$. Consider the graph $G'[S \cup X]$ and partition the vertices of X with respect to their X-neighborhoods in this graph. As $G'[S \cup X] \in G \triangledown 1 \subseteq \mathcal{G} \triangledown 1$, by Lemma 2.17 we infer that the number of these classes is at most $f^1_{\mathrm{nei}}(\delta) \cdot |X| \cdot n^{\delta}$. Hence, one of the classes, say κ, has cardinality at least c'. Since each member of $\kappa \subseteq S'$ has at least c' neighbors in X in graph G', and this X-neighborhood is common among the vertices of κ, we infer that $|N'_X(\kappa)| \geqslant c'$ and $G'[\kappa \cup N'_X(\kappa)]$ contains a biclique $K_{c',c'}$ as a subgraph. This is a contradiction with $G' \in \mathcal{G} \triangledown 1$. □

Again, we remove from S all the vertices that have X-neighborhoods in G' larger of size at least c'. In this manner, Lemma 4.4 ensures us that the size of S shrinks by at most $c' \cdot f^1_{\mathrm{nei}}(\delta) \cdot |X| \cdot n^\delta$. Hence, if we set $C := C_0 + c' \cdot f^1_{\mathrm{nei}}(\delta)$ for some C_0 to be determined later, then after performing this step we still have that the resulting set has size more than $C_0 \cdot |X| \cdot n^\delta$. Again, we shall abuse the notation and denote the resulting set also S, and we reconstruct the graph G' according to the new definition of S. In this manner, from now on we assume that $|S| > C_0 \cdot |X| \cdot n^\delta$ and that $|N'_X(s)| < c'$ for each $s \in S$.

We define the equivalence relation $\simeq_X$ over $S \cup R_2$ as before:

$$u \simeq_X v \Leftrightarrow N'_X(u) = N'_X(v).$$

Again, we shall consider quotients $K_1 := S/\simeq_X$ and $K_2 := R_2/\simeq_X$. We adopt the same notation as in Section 3: For $\kappa \in K_1 \cup K_2$, we denote by $N'_X(\kappa)$ the common X-neighborhood in G' of vertices of κ, and for $\kappa \in K_1$, we denote $U_\kappa := \bigcup_{s \in \kappa} N[s] \cap R$. As before, since X is a Z-dominator, we have that $N'_X(\kappa)$ is nonempty for each $\kappa \in K_1$. In case K_2 contains some class κ with $N'_X(\kappa) = \emptyset$, then this class is denoted by $\kappa_\emptyset$.

The definition of the class graph H is exactly the same as in Section 3: the vertex set of H is equal to $K_1 \cup K_2$, and we put $\kappa\kappa' \in E(H)$ if and only if there exist $u \in \kappa$ and $u' \in \kappa'$ such that $uu' \in E(G')$. We now prove the analogues of Lemmas 3.7 and 3.8 that estimate the size and sparsity of the class graph; the proofs are direct generalizations of the bounded expansion case to the nowhere dense case. In the following, we denote $\gamma = \delta/2$.

Lemma 4.5 (Size of the class graph)**.** *The following holds:*

- $|K_1| \leqslant f^1_{nei}(\gamma) \cdot |X| \cdot n^\gamma$, *and*
- $|K_2| \leqslant f_{nei}(\gamma) \cdot |X| \cdot n^\gamma$.

Consequently, $|V(H)| = |K_1| + |K_2| \leqslant 2f^1_{nei}(\gamma) \cdot |X| \cdot n^\gamma$.

Proof. The upper bound on $|K_2|$ (second item) follows directly from Lemma 2.17 applied to the graph $G - R_1$, set X, and parameter γ (we use that $|X| \leqslant n$). In order to obtain an upper bound on $|K_1|$, we apply Lemma 2.17 to the graph $G'[S \cup X] \in \mathcal{G} \mathbin{\triangledown} 1$. We thus infer that the number of possible X-neighborhoods among the vertices in S, and hence the number of classes in K_1, is at most $f^1_{\mathrm{nei}}(\gamma) \cdot |X| \cdot n^\gamma$. ☐

Lemma 4.6 (Grad of the class graph)**.** *There exists a function $h(\cdot)$ such that for every $r \geqslant 0$ it holds that $\nabla_r(H) \leqslant h(r) \cdot n^\gamma$.*

Proof. Let us fix r and let $\beta = \frac{\gamma}{3(3r+2)^2}$. As in the proof of Lemma 3.8, we assume that $\kappa_\emptyset$ exists; otherwise the argument is even simpler as we do not need to consider this class separately.

We construct sets L_1 and L_2 by picking an arbitrary vertex from each class of K_1 and each class of $K_2 \setminus \{\emptyset\}$, respectively. Bipartite graphs G'_1 and G'_2 are again defined as $G'_1 = (L_1, X, E(G') \cap (L_1 \times X))$ and $G'_2 = (L_2, X, E(G') \cap (L_2 \times X))$. By the definitions of classes of K_1 and K_2, and the fact that X is a Z-dominator, in the same manner as in Lemma 3.8 we infer that both these graphs satisfy the assumptions of Lemma 2.18.

Hence, Lemma 2.18 ensures us that there exist assignments $\phi_1 \colon L_1 \to X$ and $\phi_2 \colon L_2 \to X$ such that $|\phi_t^{-1}(u)| \leqslant f^1_{\mathrm{chrg}}(\beta) \cdot n^\beta$ for each $u \in X$ and $t = 1, 2$. Let us combine these assignments into $\phi \colon L_1 \cup L_2 \to X$ such that $|\phi^{-1}(u)| \leqslant \tau := 2f^1_{\mathrm{chrg}}(\beta) \cdot n^\beta$ for each $u \in X$. As in Lemma 3.8, we regard ϕ also as an assignment with domain $K_1 \cup K_2 \setminus \{\kappa_\emptyset\}$ in a natural way.

Let us consider the lexicographic product $G'' := G' \bullet K_\tau$ and its 1-shallow minor H' constructed as in Lemma 3.8: for every class $\kappa \in K_1 \cup K_2 \setminus \{\kappa_\emptyset\}$, we contract all the copies of all the vertices of κ onto one copy of $\phi(\kappa)$, so that every class is contracted onto a different vertex of G''. Since each vertex of X has been replaced with τ copies, and pre-images under ϕ have sizes bounded by τ, such a contraction is possible. After this contraction it is easy to see that the class graph $H - \{\kappa_\emptyset\}$ appears as a subgraph of H', as argued in the proof of Lemma 3.8. Hence, we can upper bound the grads of $H - \{\kappa_\emptyset\}$ using Proposition 2.4 and Lemma 2.7:

$$\begin{aligned}\nabla_r(H - \{\kappa_\emptyset\}) &\leqslant \nabla_{3r+1}(G'') \leqslant 4(8\tau(3r+1+\tau)\nabla_{3r+1}(G') + 4\tau)^{(3r+2)^2} \\ &\leqslant 4(16 f^1_{\mathrm{chrg}}(\beta)(3r+1+2f^1_{\mathrm{chrg}}(\beta))n^{2\beta} \cdot \nabla_{9r+4}(G) + 8f^1_{\mathrm{chrg}}(\beta)\cdot n^\beta)^{(3r+2)^2} \\ &\leqslant 4(24 f^1_{\mathrm{chrg}}(\beta)(3r+1+2f^1_{\mathrm{chrg}}(\beta))n^{2\beta} \cdot f_\nabla(\beta, 9r+4)\cdot n^\beta)^{(3r+2)^2} \\ &= 4(24 f^1_{\mathrm{chrg}}(\beta)(3r+1+2f^1_{\mathrm{chrg}}(\beta)) \cdot f_\nabla(\beta, 9r+4))^{(3r+2)^2} \cdot n^\gamma.\end{aligned}$$

Again, H can be obtained from $H - \{\kappa_\emptyset\}$ by adding a universal vertex and then possibly removing some edges. Hence, by Lemma 2.3 we infer that

$$\nabla_r(H) \leqslant \nabla_r(H - \{\kappa_\emptyset\}) + 1 \leqslant 5(24 f^1_{\mathrm{chrg}}(\beta)(3r+1+2f^1_{\mathrm{chrg}}(\beta)) \cdot f_\nabla(\beta, 9r+4))^{(3r+2)^2} \cdot n^\gamma.$$

This concludes the proof. □

Corollary 4.7. *There exists a constant C_E such that $|E(H)| \leqslant C_E \cdot |X| \cdot n^\delta$.*

Proof. Since $|E(H)| \leqslant \nabla_0(H) \cdot |V(H)|$, we apply the upper bounds proven in Lemma 4.5 and in Lemma 4.6 for $\alpha = \gamma$ and obtain

$$\begin{aligned}|E(H)| &\leqslant h(0) \cdot n^\gamma \cdot 2f^1_{\mathrm{nei}}(\gamma) \cdot |X| \cdot n^\gamma \\ &\leqslant 2h(0) f^1_{\mathrm{nei}}(\gamma) \cdot |X| \cdot n^\delta.\end{aligned}$$

Hence, we can take $C_E := 2h(\gamma, 0) f^1_{\mathrm{nei}}(\gamma)$. □

Now is the moment when we can finally set the constant C_0 that governs how much larger X is larger compared to X; more precisely, we assumed that $|S| > C_0 \cdot |X| \cdot n^\delta$. We namely set

$$C_0 = 2^{c'} \cdot \Big((c'+1) \cdot f^1_{\mathrm{nei}}(\gamma) + 2c' C_E\Big).$$

We can now prove the analogue of Lemma 3.10 that identifies a subclass whose size is large compared to its possible interaction in H; the proof follows exactly the same lines as in the bounded expansion case.

Lemma 4.8 (Large subclass)**.** *There exists a class $\kappa \in K_1$ and a subset $\lambda \subseteq \kappa$ with the properties that every member $s \in \lambda$ has the same neighborhood $N_X(s)$ in G and*

$$|\lambda| > c' \cdot (\deg_H(\kappa) + 1) + 1.$$

Proof. As in Lemma 3.10, we define a potential function for classes $\kappa \in K_1$ as follows:

$$\Phi(\kappa) = |\kappa| - 2^{c'}(c' \cdot (\deg_H(\kappa) + 1) + 1).$$

Summing up this potential through all the classes of K_1 we obtain the following:

$$\begin{aligned}\sum_{\kappa \in K_1} \Phi(\kappa) &= \sum_{\kappa \in K_1} |\kappa| - 2^{c'} \cdot \sum_{\kappa \in K_1} (c' \cdot (\deg_H(\kappa) + 1) + 1) \\ &= |S| - 2^{c'} \cdot \left(c' \sum_{\kappa \in K_1} \deg_H(\kappa) + (c'+1)|K_1|\right).\end{aligned}$$

We now use the fact that $\sum_{\kappa \in K_1} \deg_H(\kappa) \leqslant \sum_{\kappa \in V(H)} \deg_H(\kappa) = 2|E(H)|$ and the bounds of Lemma 4.5 and Corollary 4.7:

$$\begin{aligned}\sum_{\kappa \in K_1} \Phi(\kappa) &\geqslant |S| - 2^{c'} \left(2c' \cdot C_E \cdot |X| \cdot n^\delta + (c'+1) \cdot f^1_{\mathrm{nei}}(\gamma) \cdot |X| \cdot n^\gamma\right) \\ &\geqslant |S| - 2^{c'} \left(2c' \cdot C_E \cdot |X| \cdot n^\delta + (c'+1) \cdot f^1_{\mathrm{nei}}(\gamma) \cdot |X| \cdot n^\delta\right) \\ &= |S| - C_0 \cdot |X| \cdot n^\delta > 0.\end{aligned}$$

Hence, we infer that there exists a class $\kappa \in K_1$ such that $\Phi(\kappa) > 0$. Equivalently,

$$|\kappa| > 2^{c'}(c' \cdot (\deg_H(\kappa) + 1) + 1).$$

Exactly as in the proof of Lemma 3.10, we partition vertices of κ into subclasses with respect to the neighborhoods in X in graph G. Recall that we assumed that $|N'_X(\kappa')| < c'$ for each $\kappa' \in K_1$, and $N_X(s) \subseteq N'_X(\kappa)$ for each $s \in \kappa$, so the number of these subclasses is actually less than $2^{c'}$. Hence, there exists a subclass $\lambda \subseteq \kappa$ of vertices with the same X-neighborhood in G such that $|\lambda| \geqslant |\kappa|/2^{c'} > c' \cdot (\deg_H(\kappa) + 1) + 1$. □

We now prove the bottom line: the analogue of Lemma 3.11 where we argue that every vertex of λ is an irrelevant dominatee. Again, the proof is basically the same.

Lemma 4.9. *Let z be an arbitrary vertex of λ. Then $Z \setminus \{z\}$ is still a domination core.*

Proof. Let $Z' = Z \setminus \{z\}$ and let $N_X(\lambda) \subseteq N'_X(\kappa)$ be equal to $N_X(s)$ for any $s \in \lambda$. Take any minimum-size Z'-dominator D; we need to prove that D is a dominating set of G. In the following we work in the graph G all the time.

Suppose first that $D \cap N_X(\lambda) \neq \emptyset$. Then in particular z is also dominated by D, hence D is also a Z-dominator. As $Z \supseteq Z'$, D must be a minimum-size Z-dominator, and hence also a dominating set in G since Z was a domination core.

Suppose then that $D \cap N_X(\lambda) = \emptyset$. We are going to arrive at a contradiction with the assumption that D is of minimum possible size. Since $\lambda \setminus \{z\} \subseteq S \setminus \{z\} \subseteq Z'$, vertices of $\lambda \setminus \{z\}$ need in particular to be dominated by D. Since S is 2-scattered in $G - X$, so is $\lambda \setminus \{z\}$ as well. Hence any vertex of D can dominate only at most one vertex of $\lambda \setminus \{z\}$, as none of them can be dominated from X by the assumption that $D \cap N_X(\lambda) = \emptyset$. Also, the vertices of D that dominate vertices of $\lambda \setminus \{z\}$ need to be contained in U_κ; recall that $U_\kappa := \bigcup_{s \in \kappa} N[s] \cap R$ is the set of all vertices contracted onto vertices of κ during the construction of G'. Hence, we conclude that $|D \cap U_\kappa| \geqslant |\lambda \setminus \{z\}| > c' \cdot (\deg_H(\kappa) + 1)$.

Construct now a set D' from D by (a) removing all the vertices of $D \cap U_\kappa$, (b) adding all the vertices of $N'_X(\kappa_1)$ for every $\kappa_1 \in N_H[\kappa] \cap K_1$, and (c) adding an arbitrary vertex of $N_X(\kappa_2)$ for each $\kappa_2 \in N_H[\kappa] \cap K_2$, provided that $N_X(\kappa_2)$ is non-empty. In step (a) we have removed more than $c' \cdot (\deg_H(\kappa) + 1)$ vertices from D, whereas in steps (b) and (c) we have added in total at most $c' \cdot (\deg_H(\kappa) + 1)$ vertices: at most c' vertices per each $\kappa_1 \in N_H[\kappa] \cap K_1$, and at most one vertex per each $\kappa_2 \in N_H[\kappa] \cap K_2$. Hence, $|D'| < |D|$ and to arrive at a contradiction it remains to prove that D' is a Z'-dominator. From here on the proof is exactly the same as in Lemma 3.11, but we recall it briefly for completeness.

Take any $u \in Z'$ which became not dominated when $D \cap U_\kappa$ was removed during the construction of D'; we prove that u is dominated by the vertices added to D' in steps (b) and (c). Since u was dominated by a vertex from $D \cap U_\kappa$, we have four cases: u can belong (a) to $N'_X(\kappa)$, or (b) to U_κ, or (c) to U_{κ_1} for some $\kappa_1 \in N_H(\kappa) \cap K_1$, or (d) to some $\kappa_2 \in N_H(\kappa) \cap K_2$. Moreover, since $u \in Z'$ and X is a Z-dominator, we infer that u has at least one neighbor in X. In case (a) we have explicitly included $N'_X(\kappa)$ to D', so even $u \in D'$. In cases (b) and (c) we have added the sets $N'_X(\kappa_1)$ to D'

for each $\kappa_1 \in N_H[\kappa] \cap K_1$, so any neighbor of u in X belongs to D' and thus dominates u. In case (d), we have that $N_X(u) = N_X(\kappa_2)$ and this set is non-empty, since u indeed has a neighbor in X. Hence, we added one vertex of $N_X(u)$ to set D' and this vertex thus dominates u. □

We now conclude the proof of Lemma 4.2, which also concludes the proof of Theorem 4.1. Adopting the notation of Section 4.1.1, we take $g(\varepsilon) = (c \cdot f_{\mathrm{nei}}(\varepsilon/2) + 1) \cdot C_c$ (note that C_c also depends on ε), so that Lemma 4.3 is applicable whenever $|Z| > g(\varepsilon) \cdot k \cdot n^{\varepsilon}$. Hence, we can safely apply the algorithm of Section 4.1.1, which clearly works in polynomial time as it boils down to a constant number of applications of the algorithm of Lemma 2.19, and obtain a pair (X, S) that can be used in the second phase. Construction of the class graph H can be clearly done in polynomial time. Also, in polynomial time we can recognize the class κ and subclass $\lambda \subseteq \kappa$ that satisfy the statement of Lemma 4.8: this requires iterating through all the classes $\kappa \in K_1$, and then examining the partition of the vertices of the found class κ with respect to the neighborhoods in X. Finally, Lemma 4.9 ensures that any vertex of λ can be output by the algorithm as an irrelevant dominatee.

4.2 Reducing dominators

Having presented how to compute a small dominating core in the nowhere dense case, we can proceed to the proof of Theorem 1.2. Before this, we prove one more lemma from which the main result for the nowhere dense case will follow very easily. Its proof is essentially the same as the proof of Theorem 1.1.

Lemma 4.10. *Let $\mathcal{G}$ be a nowhere dense graph class and let $\varepsilon > 0$ be a real number. There exists a constant C_ε and a polynomial-time algorithm that, given an n-vertex graph $G \in \mathcal{G}$ and an integer k, either correctly concludes that $\mathbf{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size at most $C_\varepsilon \cdot k \cdot n^{\varepsilon}$ with the property that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$.*

Proof. The algorithm works as follows. First, using the algorithm of Theorem 3.2 for parameter $\varepsilon/2$ we compute a domination core $Z \subseteq V(G)$ such that $|Z| \leqslant g(\varepsilon/2) \cdot k \cdot n^{\varepsilon/2}$. If the algorithm of Theorem 3.2 concluded that $\mathbf{ds}(G) > k$, then we can also terminate and provide this outcome. Hence, from now on we assume that the domination core Z has been successfully computed.

Let $R := V(G) \setminus Z$ and partition the vertices of Z into classes with respect to their neighborhoods in Z. From Lemma 2.17 we infer that the number of these classes is at most $f_{\mathrm{nei}}(\varepsilon/2) \cdot |Z| \cdot n^{\varepsilon/2} \leqslant f_{\mathrm{nei}}(\varepsilon/2) g(\varepsilon/2) \cdot k \cdot n^{\varepsilon}$. Construct set Y by taking Z and, for every nonempty class κ of the considered partition, adding an arbitrarily picked vertex $v_\kappa \in \kappa$. Note that in this manner we have that:

$$|Y| \leqslant |Z| + f_{\mathrm{nei}}(\varepsilon/2) g(\varepsilon/2) \cdot k \cdot n^{\varepsilon} \leqslant (f_{\mathrm{nei}}(\varepsilon/2) + 1) g(\varepsilon/2) \cdot k \cdot n^{\varepsilon},$$

which means that we can set $C_\varepsilon := (f_{\mathrm{nei}}(\varepsilon/2) + 1) g(\varepsilon/2)$. We are left with verifying that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$. The argumentation is exactly the same as in Theorem 1.1, but we recall it for the sake of completeness.

Suppose first that $\mathbf{ds}(G) \leqslant k$, and let D be a minimum-size dominating set in G so that $|D| = \mathbf{ds}(G) \leqslant k$. As D is a dominating set in G, it is in particular a Z-dominator, and it is a minimum-size Z-dominator in G since Z is a domination core and $\mathbf{ds}(G) = \mathbf{ds}(G, Z)$. Construct set D' from D by replacing the set $\kappa \cap D$ with $\{v_\kappa\}$ for each class κ of the partition with $|\kappa \cap D| \geqslant 1$. Clearly, $|D'| \leqslant |D|$. Moreover, observe that set D' is also a Z-dominator in G, since every vertex v_κ dominates exactly the same set of vertices in Z as other vertices of κ. As D was a minimum-size Z-dominator, we infer that in fact $|D'| = |D| = \mathbf{ds}(G, Z)$ and D' is also a minimum-size Z-dominator. Since Z is a domination core, we infer that D' is a dominating set in G. Finally, as $D' \subseteq Y$, we infer that D' is also a dominating set in $G[Y]$ and hence $\mathbf{ds}(G[Y]) \leqslant |D'| \leqslant k$.

Suppose now that $\mathbf{ds}(G[Y]) \leqslant k$, and let D' be a dominating set in $G[Y]$ such that $|D'| \leqslant k$. Set D' in particular dominates the whole set $Z \subseteq Y$, which means that D' is also a Z-dominator in G. Hence $\mathbf{ds}(G, Z) \leqslant |D'| \leqslant k$. As Z is a domination core, we have that $\mathbf{ds}(G) = \mathbf{ds}(G, Z)$ and we conclude that $\mathbf{ds}(G) \leqslant k$. □

Theorem 1.2. *Let $\mathcal{G}$ be a nowhere dense graph class and let $\varepsilon > 0$ be a real number. There exists a polynomial-time algorithm that given a graph $G \in \mathcal{G}$ and an integer k, either correctly concludes that $\mathbf{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size $\mathcal{O}(k^{1+\varepsilon})$ with the property that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$.*

Proof. We apply the algorithm of Lemma 4.10 iteratively to obtain sets $V(G) = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots$: In the i-th iteration we apply the algorithm to $G[Y_{i-1}]$ in order to compute $Y_i \subseteq Y_{i-1}$. We proceed in this manner up to the point when the algorithm returns $Y_i = Y_{i-1}$, in which case we simply output $Y := Y_i$. Clearly, Y computed in this manner satisfies the requirement that $\mathbf{ds}(G) \leqslant k$ if and only if $\mathbf{ds}(G[Y]) \leqslant k$, so it remains to establish the upper bound on the size of Y.

Since the algorithm of Lemma 4.10 returned $Y_i = Y_{i-1}$, it follows that

$$|Y| = |Y_i| \leqslant C_\varepsilon \cdot k \cdot |Y_{i-1}|^\varepsilon = C_\varepsilon \cdot k \cdot |Y|^\varepsilon.$$

Here, C_ε is the constant from the statement of Lemma 4.10. Consequently,

$$|Y| \leqslant (C_\varepsilon \cdot k)^{\frac{1}{1-\varepsilon}} \leqslant C_\varepsilon^{1+2\varepsilon} \cdot k^{1+2\varepsilon}.$$

By rescaling ε by factor 2 we obtain the result. □

5 Hardness of Connected Dominating Set

In this section we prove Theorem 1.4; let us recall its statement.

Theorem 1.4. *There exists a class of graphs $\mathcal{G}$ of bounded expansion such that* Connected Dominating Set *does not admit a polynomial kernel when restricted to $\mathcal{G}$, unless* NP $\subseteq$ coNP/poly*, and furthermore, $\mathcal{G}$ is closed under taking subgraphs.*

The proof of Theorem 1.4 is a refinement of the proof of Cygan et al. [6] that Connected Dominating Set does not admit a polynomial kernel in graphs of bounded degeneracy. The main idea of [6] is to use Graph Motif as a pivot problem.

Graph Motif	**Parameter:** k
Input: A graph G, an integer k, and a surjective function $c : V(G) \to [k]$.	
Question: Does there exist a set $X \subseteq V(G)$ of size exactly k such that $G[X]$ is connected and $c|_X$ is bijective?	

We call the function c a *coloring* and each value $i \in [k]$ is a *color*. In this wording, in the Graph Motif problem we seek for a set of vertices, one of every color, that induces a connected subgraph of G.

Fellows et al. [12] were first to study the parameterized complexity of Graph Motif and, among other results, they prove that the problem is hard already in a very restrictive setting.

Theorem 5.1 ([6,12])**.** *The* Graph Motif *problem, restricted to graphs G being trees of maximum degree 3, is NP-complete and does not admit a polynomial compression when parameterized by k unless* NP $\subseteq$ coNP/poly.

Here, a *polynomial compression* is a generalization of the notion of a polynomial kernel, where we relax the requirement that the output needs to be an instance of a original problem. Formally, a polynomial compression from a parameterized language P into a (classic) language L is an algorithm that, given an instance (x, k), works in time polynomial in $|x| + k$ and outputs a string y with the following properties: (i) $(x, k) \in P$ if and only if $y \in L$, and (ii) $|y|$ is bounded polynomially in k.

The main observation of [6] is that Graph Motif easily reduces to Connected Dominating Set. Let $I = (G, k, c)$ be a Graph Motif instance. Consider a graph G_I^{cds} constructed as follows: we first take $G_I^{\text{cds}} = G$ and then, for every color $i \in [k]$, we add two vertices w_i and w_i°, connected by an edge, and make w_i adjacent to $c^{-1}(i)$, that is, to all vertices of G of color i. It is easy to observe the following.

Lemma 5.2 ([6])**.** *I is a yes-instance to* Graph Motif *if and only if G_I^{cds} admits a connected dominating set of size at most $2k$.*

Proof. Let $W = \{w_i : 1 \leqslant i \leqslant k\}$. Observe that W is a dominating set in G_I^{cds}. If $k = 1$, then W is also connected and the claim is trivial, so assume $k \geqslant 2$.

In one direction, observe that if X is a solution to Graph Motif instance, then $X \cup W$ is a connected dominating set in G of size $2k$: W dominates $V(G)$, while $G[X]$ is connected and every $w_i \in W$ has a (unique) neighbor in $X \cap c^{-1}(i)$.

In the other direction, let Y be a connected dominating set of size at most $2k$ in G_I^{cds}. Observe that, due to pendant vertices w_i°, the set Y needs to contain W. Since $k \geqslant 2$, to make $W \subseteq Y$ connected, for every $1 \leqslant i \leqslant k$ the set Y needs to contain a vertex $y_i \in c^{-1}(i)$. Since $|Y| \leqslant 2k$, we have already enumerated all vertices of Y: $Y = \{w_i : 1 \leqslant i \leqslant k\} \cup \{y_i : 1 \leqslant i \leqslant k\}$. Thus, every w_i is of degree one in $G_I^{\text{cds}}[Y]$ and, consequently, $G_I^{\text{cds}}[\{y_i : 1 \leqslant i \leqslant k\}]$ is connected. Hence, $\{y_i : 1 \leqslant i \leqslant k\}$ is a solution to Graph Motif on I. ☐

In is easy to see that the reduction from $I = (G, k, c)$ to G_I^{cds} described above translates not only NP-hardness, but also kernelization lower bound: any polynomial compression for Connected Dominating Set, pipelined with the aforementioned reduction, would give a polynomial compression for Graph Motif.

As observed in [6], if G is a tree, then G_I^{cds} is 2-degenerate. However, G_I^{cds} may not be of bounded expansion, due to arbitrary connections in the graph introduced by the edges incident to vertices w_i. Our main goal for the rest of this section is to tweak the reduction described above to make G_I^{cds} of bounded expansion.

To control the expansion of G_I^{cds} — and prove Theorem 1.4 — we need to control how the colors of I can neighbor each other. More formally, given an instance $I = (G, k, c)$ of Graph Motif, let us define the *color graph* H_I^{col} to be a graph with vertex set $V(H_I^{\text{col}}) = [k]$ and $ij \in E(H_I^{\text{col}})$ if and only if there exists an edge $xy \in E(G)$ with $c(x) = i$ and $c(y) = j$. The next lemma shows that if we can control the maximum degree of H_I^{col}, then G_I^{cds} is of bounded expansion.

Lemma 5.3. *Let (G, k, c) be a* Graph Motif *instance. Assume that the maximum degree of G is at most Δ_G, and the maximum degree of H_I^{col} is at most Δ_H. Then, for every $r \geqslant 1$, every r-shallow topological minor of G_I^{cds} is* $\max(\Delta_G + 1, (\Delta_H + 1)^{2r})$*-degenerate.*

Proof. Fix $r \geqslant 1$. Let H be an r-shallow topological minor of G_I^{cds}. To prove the lemma, it suffices to show that H contains a vertex of degree at most $\max(\Delta_G + 1, (\Delta_H + 1)^{2r} + 1)$; the same reasoning can be performed for every induced subgraph of H. Let us fix one model of H in G_I^{cds}, and consider one vertex $x \in V(H)$ mapped to a root vertex $v \in V(G_I^{\text{cds}})$.

If $v \in V(G)$, then the degree of v in G_I^{cds} is at most $\Delta_G + 1$, and the same bound holds for the degree of x in H. If $v = w_i^\circ$ for some $1 \leqslant i \leqslant k$, then the degree of v in G_I^{cds} is 1, and the degree of

x in H is at most 1. Thus, it remains to consider the case where every vertex $x \in V(H)$ is mapped to some vertex w_i, $1 \leqslant i \leqslant k$.

Consider then a vertex w_i. For an integer $d \geqslant 1$, we say that a color j is *reachable within distance* d *from* w_i if there exists a vertex $v \in V(G_I^{\mathrm{cds}})$ within distance d from w_i such that $c(v) = j$. Let L_d be the set of colors reachable from w_i within distance d. Observe that the bound on the degree of H_I^{col} implies the following:

Claim 5.4. *For every $d \geqslant 1$ it holds that $|L_d| \leqslant (\Delta_H + 1)^{d-1}$.*

Proof. We prove by induction on d. For $d = 1$, observe that $L_1 = \{i\}$.

Consider now $j \in L_{d+1} \setminus L_d$. Since $j \notin L_d$ and every vertex w_ι has only neighbors in $c^{-1}(\iota)$ (apart from the pendant w_ι°), there exists a color $j' \in L_d$ and an edge $xy \in E(G)$ such that $c(x) = j$ and $c(y) = j'$. Consequently, $jj' \in E(H_I^{\mathrm{col}})$. Since the maximum degree of H_I^{col} is bounded by Δ_H, we have $|L_{d+1} \setminus L_d| \leqslant \Delta_H |L_d|$ and the claim follows. ⌟

By Claim 5.4, for a fixed vertex w_i, at most $(\Delta_H + 1)^{2r}$ other vertices w_j are within distance at most $2r + 1$ in G_I^{cds} from w_i. Consequently, no r-shallow topological minor with roots in vertices w_i can have a vertex of degree more than $(\Delta_H + 1)^{2r}$. This concludes the proof of the lemma. □

By Lemma 5.3, to prove Theorem 1.4 it suffices to show that the lower bounds of Theorem 5.1 still hold if we restrict the maximum degree of H_I^{col}. Luckily, this turns out to be quite an easy task (see also Figure 4 for an illustration of the gadget used).

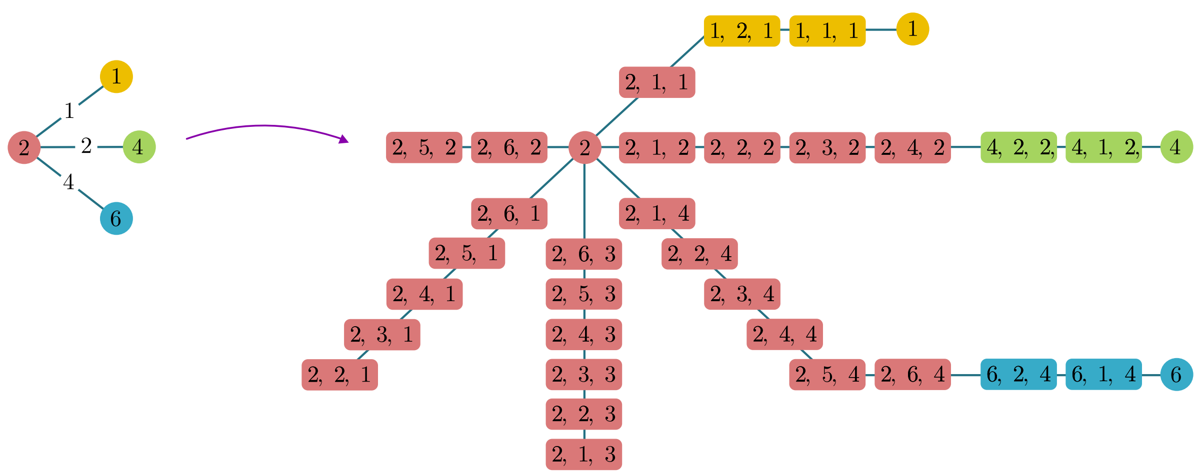

Figure 4: Part of the graph corresponding to a vertex u of color $c(u) = 2$, with neighbors of colors 1, 4 and 6, and assuming $k = 6$ and $\Delta_G = 3$. The numbers on edges correspond to their colors in the coloring f.

Lemma 5.5. *There exists a polynomial algorithm that, given a* Graph Motif *instance (G, k, c) where the maximum degree of G is bounded by Δ_G, outputs an equivalent* Graph Motif *instance $I' = (G', k', c')$ where $k' = k + (\Delta_G + 1)k^2$, the maximum degree of G' is bounded by $2\Delta_G + 2$, and the maximum degree of $H_{I'}^{col}$ is bounded by $\max(2\Delta_G + 2, 3)$.*

Proof. For clarity of presentation, we identify the new set of colors, $[k']$, with $[k] \cup ([k] \times [k] \times [\Delta_G + 1])$. By Vizing's theorem, the edges of G can be colored with $\Delta_G + 1$ colors such that no two incident edges have the same color. Moreover, such a coloring can be found in polynomial time [24]. Let $f : E(G) \to [\Delta_G + 1]$ be any such coloring.

For integers $i, a, b \in [k]$ and $\alpha \in [\Delta_G + 1]$ with $a \leqslant b$ we define an $(i, \alpha; a, b)$-path to be a path on $b - a + 1$ vertices denoted $x_{i,j,\alpha}$ for $a \leqslant j \leqslant b$ and with colors $c'(x_{i,j,\alpha}) = (i, j, \alpha)$.

We construct the instance I' as follows. We start with $V(G') = V(G)$ and $c' = c$. Then, for every edge uv we make the following construction. Assume $c(u) = i$, $c(v) = j$, and $f(uv) = \alpha$. We first take an $(i, \alpha; 1, j)$-path P_u^α and an $(j, \alpha; 1, i)$-path P_v^α, and connect them as follows: we make $x_{i,1,\alpha}$ on P_u^α adjacent to u, $x_{j,1,\alpha}$ on P_v^α adjacent to v, and $x_{i,j,\alpha}$ on P_u^α adjacent to $x_{j,i,\alpha}$ on P_v^α. In this way we have added a path $P_u^\alpha \cup P_v^\alpha$ between u and v of length $j + i + 1$. Second, if $j < k$ we take a $(i, \alpha; j + 1, k)$-path Q_u^α and make $x_{i,k,\alpha}$ on this path adjacent to u. Similarly, if $i < k$ we take a $(j, \alpha; i + 1, k)$-path Q_v^α and make $x_{j,k,\alpha}$ on this path adjacent to v. If $j = k$ or $i = k$, then the corresponding path Q_u^α or Q_v^α is defined to be an empty path for the sake of further notation.

Furthermore, if for some $u \in V(G)$ and $\alpha \in [\Delta_G + 1]$ there does not exist an edge incident to u colored (by f) with color α, then we create a $(c(u), \alpha; 1, k)$-path Q_u^α and make $x_{i,k,\alpha}$ on this path adjacent to u.

This concludes the description of the instance $I' = (G', k', c')$. In the next three claims we prove the desired properties of I'.

Claim 5.6. *The instances I and I' are equivalent.*

Proof. For $u \in V(G)$, let W_u be the set of vertices of G' associated with u, that is, the vertex u as well as all vertices on all paths P_u^α and Q_u^α, $\alpha \in [\Delta_G + 1]$. Observe that, by construction, the set W_u contains exactly one vertex of every color of $\{c(u)\} \cup (\{c(u)\} \times [k] \times [\Delta_G + 1])$, and no vertices of other colors. Furthermore, $G[W_u]$ is connected. Consequently, if $X \subseteq V(G)$ is a solution to the Graph Motif instance I, then $X' := \bigcup_{u \in X} W_u$ is a solution to I': for every edge $uv \in E(G[X])$, the corresponding path $P_u^{f(uv)} \cup P_v^{f(uv)}$ is completely contained in $G'[X']$.

In the other direction, let $X' \subseteq V(G')$ be a solution to I'. We claim that $X := X' \cap V(G)$ is a solution to I. If $k = 1$, then the claim is trivial, so assume $k \geqslant 2$. Clearly, X contains exactly one vertex of every color of $[k]$. Consider the following graph G_X: $V(G_X) = X$ and $uv \in E(G_X)$ if and only if there exists a path in $G'[X']$ between u and v with no internal vertex in X. Clearly, the connectivity of $G'[X']$ implies that G_X is connected as well. Furthermore, observe that every vertex of $V(G') \setminus V(G)$ in G' is of degree at most 2. Consequently, for every $uv \in E(G_X)$, the corresponding path in $G'[X']$ has to be equal to $P_u^{f(uv)} \cup P_v^{f(uv)}$; in particular, $uv \in E(G)$. We infer that G_X is a subgraph of $G[X]$ and, hence, $G[X]$ is connected. This finishes the proof of the claim. ⌟

Claim 5.7. *The maximum degree of G' is at most $2\Delta_G + 2$.*

Proof. Every vertex of $V(G') \setminus V(G)$ is of degree at most two in G'. Every vertex $v \in V(G)$ is adjacent in G' to at most one vertex of every color of $\{c(v)\} \times \{1, k\} \times [\Delta_G + 1]$, and thus is of degree at most $2(\Delta_G + 1)$. ⌟

Claim 5.8. *The maximum degree of $H_{I'}^{col}$ is at most $\max(3, 2\Delta_G + 2)$.*

Proof. As already observed, every vertex $v \in V(G)$ is adjacent to at most one vertex of every color of $\{c(v)\} \times \{1, k\} \times [\Delta_G + 1]$. Thus, the degree of the color $i \in [k]$ in $H_{I'}^{\mathrm{col}}$ is at most $2(\Delta_G + 1)$. Furthermore, observe that a vertex of color $(i, j, \alpha) \in [k] \times [k] \times [\Delta_G + 1]$ can be adjacent only to vertices of colors: (j, i, α), $(i, j + 1, \alpha)$ if $j < k$, $(i, j - 1, \alpha)$ if $j > 1$, and i if $j \in \{1, k\}$. Thus, the degree of the color (i, j, α) in $H_{I'}^{\mathrm{col}}$ is at most 3. ⌟

The above three claims conclude the proof of Lemma 5.5. □

Lemma 5.5 translates the lower bounds of Theorem 5.1 to the case of bounded degree of H_I^{col}, by setting $\Delta_G = 3$.

Corollary 5.9. *The* Graph Motif *problem, restricted to instances $I = (G, k, c)$ where the maximum degree of G and the maximum degree of H_I^{col} is at most 8, is NP-complete and does not admit a polynomial compression when parameterized by k unless* NP $\subseteq$ coNP/poly.

Let us conclude with a wrap up of the proof of Theorem 1.4. Let $\mathcal{G}$ be the class of graphs where, for every $r \geqslant 1$, every r-shallow topological minor is 9^{2r}-degenerate. Assume we have a polynomial compression algorithm $\mathcal{A}$ for Connected Dominating Set restricted to $\mathcal{G}$. Let $I = (G, k, c)$ be a Graph Motif instance where the maximum degree of G and the maximum degree of H_I^{col} is at most 8. By Lemma 5.3, $G_I^{\mathrm{cds}} \in \mathcal{G}$. Thus, by applying $\mathcal{A}$ to G_I^{cds} for every such instance I, we obtain a polynomial compression for Graph Motif for instances with the maximum degree of G and H_I^{col} bounded by 8. Theorem 1.4 follows then from Corollary 5.9.

6 Conclusions and Further Research

In this paper we have presented the first linear kernel for Dominating Set on graph classes of bounded expansion, and the first almost linear linear kernel for the problem on nowhere dense graph classes. We would like to point out several features of our algorithm that at first glance may be not apparent from its description.

First of all, in our proofs we did not use the full power of nowhere dense and bounded expansion graph classes, since the whole reasoning used $\nabla_r(G)$ only for $r \leqslant 4$. Therefore, our kernelization algorithm works equally well on graph classes that have finite $\nabla_4(\mathcal{G})$. An example of such a graph class is the class of subgraphs of cliques with every edge subdivided 9 times; observe that this class is actually somewhere dense. We suspect that when fully understood, our arguments in fact use only $\nabla_2(G)$. More precisely, the term $\nabla_4(G)$ appears usually as a bound on $\nabla_1(G')$ for some $G' \in G \triangledown 1$, but the actually examined 1-shallow minor of a 1-shallow minor of G is usually not only a 4-shallow minor of G, but in fact a 2-shallow minor. However, tracing the exact shallowness of examined minors would be cumbersome, especially since the argumentation of Lemma 2.10 that uses weak colorings would need to be investigated as well. Therefore, we decided to trade the tightness for simplicity with this respect and base all our upper bounds on $\nabla_4(G)$.

Secondly, we would like to point out that the algorithm in fact does not necessarily need to have an a priori knowledge of the values of $\nabla_r(\mathcal{G})$ for $0 \leqslant r \leqslant 4$. In fact, the algorithm can be run with a hypothetical upper bound on $\nabla_4(G)$, and it will either succeed in finding a correct kernel, or it will find a proof that the actual value of $\nabla_4(G)$ is larger than assumed. Indeed, the crucial exchange argument in the proof of Lemma 3.11 only compares the actual number of vertices in the subclass λ with the actual total number of exchanged vertices in the X-neighborhoods of classes neighboring κ. Hence, whenever this comparison reveals that any member of λ is an irrelevant dominatee, this conclusion is drawn independently of the actual value of $\nabla_4(G)$, and hence is always correct. As a result, the algorithm can be run with larger and larger hypothetical bounds up to the point when a kernel is constructed. Therefore, after easy modifications the algorithm can be applied to basically *any* graph in hope of finding a reasonable kernel, and our analysis only shows guarantees on the output size in terms of the graph's densest 4-shallow minor.

Thirdly, whereas the constant in the kernel size may seem impractical, we would like to point out that it provides a major improvement over the previous works. The kernels for H-minor-free and H-topological-minor-free graphs of [16,17] are based on arguments originating in bidimensionality theory, graph minors, and finite-state properties of Dominating Set. Therefore, the dependence of the constant in the kernel size on the size of H is very difficult to trace. Even very crude estimations show that it is multiple-exponential, however still elementary. Our analysis shows that the kernel given by Theorem 1.1 has size $2^{\mathcal{O}(\nabla_0(\mathcal{G})\cdot\nabla_4(\mathcal{G}))} \cdot k$, whereas for $\mathcal{G}$ being the class of H-minor-free graphs we have that $\nabla_0(\mathcal{G}) = \nabla_4(\mathcal{G}) = \mathcal{O}(|V(H)| \cdot \sqrt{\log |V(H)|})$ [26, Lemma 4.1]. Thus, the constants obtained using our general technique are not only explicit, but also much lower than the ones obtained earlier.

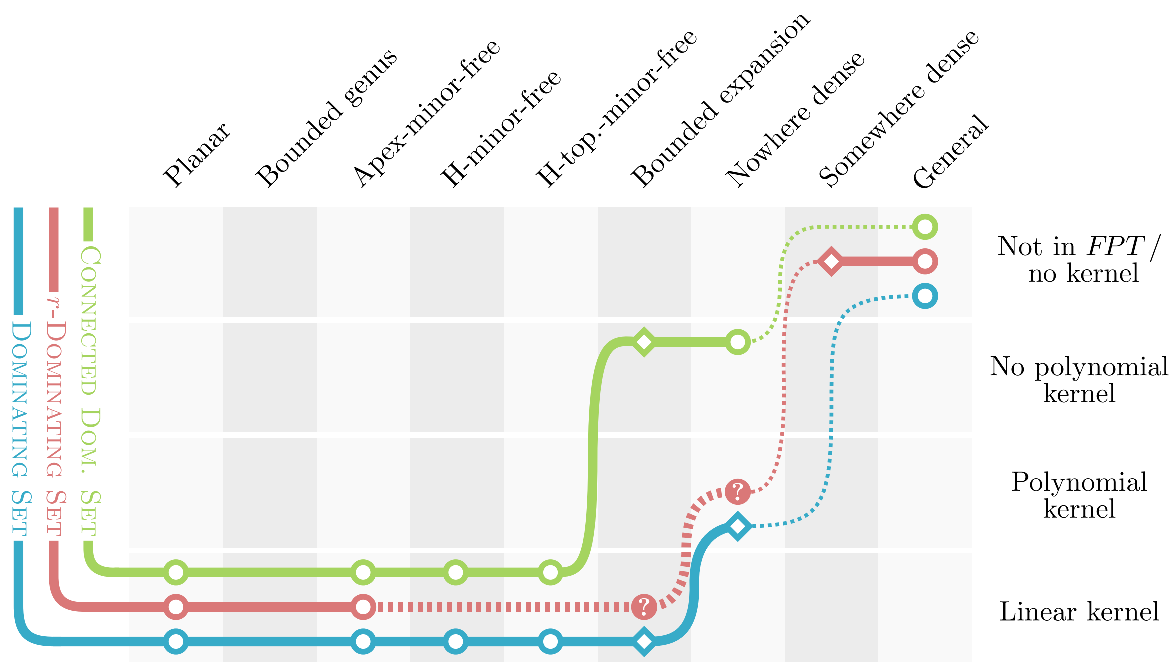

Figure 5: An overview over results contained in past publications (circles) and this paper (diamonds) for Dominating Set, r-Dominating Set and Connected Dominating Set. The dashed lines and question marks are conjectures. The dotted lines represent the (unclear) transition of the complexity between nowhere dense graph classes and general graphs through larger and larger classes of somewhere dense graphs.

The presented work leaves, however, a number of nurturing open questions and interesting prospects of future work.

In order to make the algorithm more practical it is necessary to implement it in time linear in the size of the graph. In the current presentation we have not estimated the exact running time of the kernelization procedure; however, it is at least quadratic due to removing vertices from the domination core one by one. We expect that with more technical insight, the irrelevant dominatees can be removed in larger portions, which would lead to linear running time. However, we wanted to keep the current presentation as simple as possible, and hence we deferred optimizing the running time to future work.

From the theoretical point of view, the most important question left is the complexity of r-Dominating Set, where every vertex dominates a ball of radius r around it. So far, a linear kernel for this problem has been given only for bounded genus graphs [4], and the status of its kernelization complexity is open even in H-minor-free graphs. Actually, on H-minor free graphs it is not even known whether r-Dominating Set admits a subexponential parameterized algorithm. We expect, however, that r-Dominating Set indeed admits a linear kernel on graph classes of bounded expansion and an almost linear kernel on nowhere dense graph classes.

This conjecture leads us to an interesting direction of investigating the limits of efficient kernelization of Dominating Set on sparse graph classes, mirroring the tight situation for model checking first order logic [11,21]. As we have argued, for the standard variant of this problem the nowhere dense classes are not the ultimate limit, since our algorithm works well on subgraphs of 9-times subdivided cliques, which form a somewhere dense class. Hence, we expect that the border of tractability for the standard variant of Dominating Set is unclear and difficult to grasp. However, for the more general r-Dominating Set there is no obstacle of this form. In fact, using the same technical characterization of somewhere dense classes as Dvořák et al. [11] in their proof of intractability of model checking first order logic formulae, we are able to prove the following statement:

Theorem 6.1. [$\star$] *For every somewhere dense graph class $\mathcal{G}$ that is closed under taking subgraphs, there exists an integer r such that r-*Dominating Set *is $W[2]$-hard on graphs from $\mathcal{G}$.*

Hence, it is even implausible that on a somewhere dense graph class there are FPT algorithms for all the r-Dominating Set problems, not to mention the existence of polynomial kernels. Thus, a positive result for r-Dominating Set on nowhere dense graph classes would confirm the following conjecture that we pose: A class $\mathcal{G}$ that is closed under taking subgraphs is nowhere dense if and only if for every integer $r \geqslant 1$ and real $\varepsilon > 0$, r-Dominating Set admits an $\mathcal{O}(k^{1+\varepsilon})$ kernel on $\mathcal{G}$.

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Appendix

Lemma 2.3. *Let G be a graph and let G' be obtained from G by adding a universal vertex to G, i.e., a vertex that is adjacent to every vertex of $V(G)$. Then*

$$\nabla_r(G) \leqslant \nabla_r(G') \leqslant \nabla_r(G) + 1.$$

Proof. The first inequality follows immediately from the fact that if M is an r-shallow minor of G, then M is an r-shallow minor of G'. For the second inequality, let M' be an r-shallow minor of G' with density$(M') = \nabla_r(G')$. If the minor model of M' in G' does not contain the universal vertex, we have that $\nabla_r(G') = \nabla_r(G)$. So suppose it contains the universal vertex. Then, by using the same minor model in G but removing the branch set that contains the universal vertex, we obtain an r-shallow minor M of G which lacks one vertex and at most $|M'| - 1$ edges with respect to M'. Hence, we have the following:

$$\nabla_r(G) \geqslant \text{density}(M) \geqslant \frac{||M'|| - |M'| + 1}{|M'| - 1} =$$
$$= \frac{||M'||}{|M'| - 1} - 1 \geqslant \text{density}(M') - 1 = \nabla_r(G') - 1.$$

□

The topological grad $\widetilde{\nabla}_r(G)$ of a graph G is defined similarly to the grad, but we restrict ourselves to topological r-shallow minors, i.e., we may only contract vertex disjoint paths as follows: A shallow topological minor of a given graph G at depth r, for some half-integral r, is a graph H obtained from G by taking a subgraph and then contracting internally vertex disjoint paths of length at most $2r+1$ to edges. We denote the set of r-shallow topological minors of G by $G \mathbin{\widetilde{\triangledown}} r$. Then the definition of a topological grad follows:

Definition 6.2 (Topological grad (top-grad))**.** Let G be a graph and r a half-integral. Then we define the topological grad as

$$\widetilde{\nabla}_r(G) = \max_{H \in G \mathbin{\widetilde{\triangledown}} r} \text{density}(H).$$

It is known that topological grads are comparable to normal ones; for the following inequalities see Corollary 4.1 of [26]:

$$\widetilde{\nabla}_r(G) \leqslant \nabla_r(G) \leqslant 4(4\widetilde{\nabla}_r(G))^{(r+1)^2} \tag{3}$$

Using the notion of topological grad as a pivot parameter, we can now prove Lemma 2.7.

Lemma 2.7. *For any graph G and non-negative integers $c \geqslant 1$ and r we have that*

$$\nabla_r(G \bullet K_c) \leqslant 4(8c(r+c) \cdot \nabla_r(G) + 4c)^{(r+1)^2}.$$

Proof. The following inequality has been proven in [26, Proposition 4.6]:

$$\widetilde{\nabla}_r(G \bullet K_c) \leqslant \max\{2r(c-1)+1, c^2\} \cdot \widetilde{\nabla}_r(G) + c - 1.$$

We remark that even though Proposition 4.6 of [26] assumes that $c \geqslant 2$, the claim holds also for $c = 1$. We now observe that

$$\max\{2r(c-1)+1, c^2\} \leqslant 2r(c-1) + 1 + c^2 \leqslant 2c(r+c),$$

and hence

$$\widetilde{\nabla}_r(G \bullet K_c) \leqslant 2c(r+c) \cdot \widetilde{\nabla}_r(G) + c \tag{4}$$

By combining (3) and (4) we obtain

$$\begin{aligned}\nabla_r(G \bullet K_c) &\leqslant 4(4\widetilde{\nabla}_r(G \bullet K_c))^{(r+1)^2} \\ &\leqslant 4(8c(r+c) \cdot \widetilde{\nabla}_r(G) + 4c)^{(r+1)^2} \\ &\leqslant 4(8c(r+c) \cdot \nabla_r(G) + 4c)^{(r+1)^2},\end{aligned}$$

as claimed. □

Theorem 2.11. *There is a polynomial-time algorithm that given a graph G and an integer k, either finds a dominating set of size at most $2^{20}\nabla_1(G)^{12}k$ or a 2-scattered set of size at least $k+1$ in G.*

Proof. The core argument of Dvořák [10] lies in the following statement; here $\alpha_m(G)$ denotes the maximum size of an m-scattered set in G.

Theorem 6.3 (Theorem 4 of [10])**.** *If $1 \leqslant m \leqslant 2k+1$ and G satisfies $\mathrm{wcol}_m(G) \leqslant c$, then $\mathbf{ds}(G) \leqslant c^2\alpha_m(G)$. Furthermore, if an ordering σ of $V(G)$ such that $|B^\sigma_m(v)| < c$ for every $v \in V(G)$ is given, then a k-dominating set D and an m-scattered set A such that $|D| \leqslant c^2|A|$ can be found in $\mathcal{O}(c^2 \cdot \max(k,m) \cdot |V(G)|)$ time.*

If we set $k=1$ and $m=2$, then the proof of Theorem 2.11 boils down to finding an ordering of $V(G)$ with a near-optimal 2-weak coloring number. As Dvořák observes, this can be done using the notion of m-admissibility, which is a similar measure of orderings of $V(G)$ as weak colorings. In particular (see Lemma 5 in [10] and the discussion after it), an ordering of $V(G)$ of 2-admissibility c has weak coloring number at most $(c(c-1)+1)^2$. Also, as Dvořák [10], argues m-admissibility admits a simple polynomial-time m-approximation algorithm. By applying this algorithm we can thus obtain an ordering of $V(G)$ with weak coloring number at most $(2c(2c-1)+1)^2 \leqslant 16c^2$, where $c = \mathrm{adm}_2(G)$ is the optimum 2-admissibility of G.

We are left with bounding the 2-admissibility of a graph in terms of its grads. For this, we use a trivial bound $\mathrm{adm}_2(G) < \mathrm{col}_2(G)$ (see Exercise 4.5 in [26]) and the bound

$$\mathrm{col}_2(G) \leqslant 1 + 8\nabla_1(G)^3,$$

following from Theorem 7.11 in [26]. Thus $\mathrm{adm}_2(G) \leqslant 8\nabla_1(G)^3$, and hence the approximation algorithm for 2-admissibility outputs an ordering with weak coloring number at most

$$16 \cdot (8\nabla_1(G)^3)^2 = 2^{10}\nabla_1(G)^6.$$

By applying the algorithm of Theorem 6.3 we can either find a 2-scattered set A of size at least $k+1$, or a dominating set D of size at most $2^{20}\nabla_1(G)^{12} \cdot k$, as claimed. □

Lemma 2.18. *Let $\mathcal{G}$ be a nowhere dense graph class. Then there exists a function $f_{chrg}(\cdot)$ such that the following holds. For any $\varepsilon > 0$ and any bipartite graph $G = (X, Y, E) \in \mathcal{G}$ such that every vertex from Y has a nonempty neighborhood in X and no two vertices of Y have the same neighborhood in X, there exists a mapping $\phi \colon Y \to X$ with the following properties:*

- *$u\phi(u) \in E$ for each $u \in Y$;*
- *$|\phi^{-1}(v)| \leqslant f_{chrg}(\varepsilon) \cdot |G|^\varepsilon$ for each $v \in X$.*

Proof. Without loss of generality assume that $\mathcal{G}$ is closed under taking subgraphs, since otherwise we can consider the closure of $\mathcal{G}$ under this operation, which is also nowhere dense.

We mimic the proof of Lemma 2.10. Let us fix $G = (X, Y, E)$ and $\varepsilon > 0$. Using Lemma 2.9 we infer that there exists an ordering $\sigma \in \Pi(G)$ such that for every vertex v, we have $|B_2^\sigma(v)| \leqslant (8\nabla_1(\mathcal{G})^3+1)^2$. By applying Proposition 2.14, point (1), for $r = 1$ and $\varepsilon/12$, we obtain that $|B_2^\sigma(v)| \leqslant f_0(\varepsilon) \cdot |G|^{\varepsilon/2}$, for some value $f_0(\varepsilon)$ depending on $f_\nabla(1, \varepsilon/12)$.

As in the proof of Lemma 2.10, construct ϕ by setting $\phi(u)$ to the last vertex of $N(u)$ in σ. Again, the definition is valid since Y does not contain any isolated vertices, and the first condition is trivially satisfied.

To prove the second condition, fix a vertex $v \in X$ and consider all the vertices u with $\phi(u) = v$. Let $U^- = \{u \,:\, u \in Y \wedge \phi(u) = v \wedge u <_\sigma v\}$ and $U^+ = \{u \,:\, u \in Y \wedge \phi(u) = v \wedge v <_\sigma u\}$. Similarly as in the proof of Lemma 2.10, we have that $U^- \subseteq B_1^\sigma(v)$ and hence

$$|U^-| \leqslant |B_1^\sigma(v)| \leqslant |B_2^\sigma(v)| \leqslant f_0(\varepsilon) \cdot |G|^{\varepsilon/2}.$$

Also, for all vertices $u \in U^+$ we have that $N(u) \subseteq B_2^\sigma(v) \cup \{v\}$. Since every pair of vertices in Y have different neighborhoods in X, we can apply Lemma 2.17 to the bipartite graph induced in G between $B_2^\sigma(v)$ and U^+ (note that this graph belongs to $\mathcal{G}$ since $\mathcal{G}$ is closed under taking subgraphs) and parameter 1, and conclude that

$$|U^+| \leqslant f_{\mathrm{nei}}(1) \cdot |B_2^\sigma(v)|^2 \leqslant f_{\mathrm{nei}}(1) \cdot f_0(\varepsilon)^2 \cdot |G|^{\varepsilon}.$$

Concluding,

$$|\phi^{-1}(v)| = |U^-| + |U^+| \leqslant f_0(\varepsilon) \cdot |G|^{\varepsilon/2} + f_{\mathrm{nei}}(1) \cdot f_0(\varepsilon)^2 \cdot |G|^{\varepsilon}.$$

Hence we can take $f_{\mathrm{chrg}}(\varepsilon) = f_0(\varepsilon) + f_{\mathrm{nei}}(1) \cdot f_0(\varepsilon)^2$. □

Theorem 6.1. *For every somewhere dense graph class $\mathcal{G}$ that is closed under taking subgraphs, there exists an integer r such that r-*Dominating Set *is $W[2]$-hard on graphs from $\mathcal{G}$.*

Proof. Let $\mathcal{H}_p$ be the class of p-subdivisions of all the simple graphs, that is, the class comprising all the graphs that can be obtained from any simple graph by replacing every edge by a path of length p. We need the following claim, which Dvořák et al. [11] attribute to Nešetřil and Ossona de Mendez [25]. Unfortunately, in [25] we could not find the proof of this exact statement, so for the sake of completeness we prove it ourselves.

Claim 6.4. *For every somewhere dense graph class $\mathcal{G}$ that is closed under taking subgraphs, there exists an integer r_0 such that $\mathcal{H}_{r_0} \subseteq \mathcal{G}$.*

Proof. Since $\mathcal{G}$ is somewhere dense, by [25, Theorem 4.1 (iii)] we have there exists a constant r_1 such that $\mathcal{G}$ contains every complete graph as a topological minor of depth r_1. Since $\mathcal{G}$ is closed under taking subgraphs, this means that for every $n \in \mathbb{N}$ there exists a graph $H_n \in \mathcal{G}$ that can be obtained from a clique K_n by replacing every edge by a path of length at most $r_2 := 2r_1 + 1$.

For every n, let $N(n)$ be the Ramsey number such that a complete graph on $N(n)$ vertices with edges colored with r_2 colors always contains a monochromatic complete subgraph on n vertices. Examine the complete graph $K_{N(n)}$ and assign to every edge of $K_{N(n)}$ a color from $\{1, 2, \ldots, r_2\}$ depending on the length of the corresponding path in $H_{N(n)}$. By the definition of $N(n)$ we infer that there exists a color $r(n) \in \{1, 2, \ldots, r_2\}$ such that there is a monochromatic complete subgraph on n vertices with every edge colored with $r(n)$. This means that $H_{N(n)}$ contains a subgraph that is isomorphic to clique K_n with every edge replaced by a path of length $r(n)$. Thus, $H_{N(n)}$ contains as subgraphs also all the $r(n)$-subdivisions of all the graphs on at most n vertices. We conclude by taking r_0 to be any number that appears infinitely many times in the sequence $(r(i))_{i \in \mathbb{N}}$. ⌟

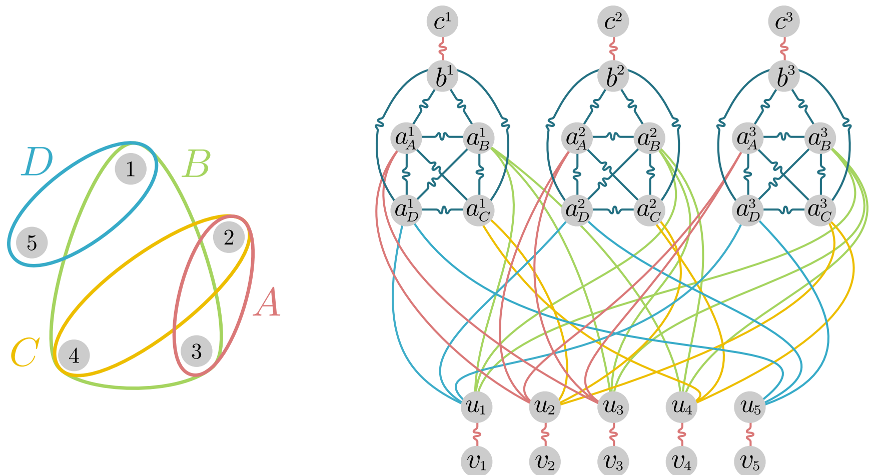

Figure 6: Example of the reduction for $U = [5], \mathcal{F} = \{A, B, C, D\}$ and $k = 3$. The edges on the right denote paths of length $2r_0$, except those connecting b_i, c_i and u_e, v_e whose length is r_0.

By Claim 6.4, the proof of Theorem 6.1 reduces to proving that for any integer $r_0 \geqslant 0$ there exists an integer r such that r-Dominating Set is W[2]-hard on the class $\mathcal{H}_{r_0}$. We prove this fact for $r = 3r_0$ by a reduction from the Set Cover problem parameterized by the requested solution size, which is known to be W[2]-hard [9,14]. Recall that the instance of the Set Cover problem consists of $(U, \mathcal{F}, k)$, where U is a finite universe, $\mathcal{F} \subseteq 2^U$ is a family of subsets of the universe, and k is an integer. The question is whether there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size k such that every element of U is covered by $\mathcal{G}$, i.e., $\bigcup \mathcal{G} = U$.

Given the instance $(U, \mathcal{F}, k)$, we construct a graph G as follows; see Figure 6 for an illustration. First, for every $i \in [k]$ do the following:

- For each $X \in \mathcal{F}$, create a vertex a^i_X; let $A^i = \{a^i_X : X \in \mathcal{F}\}$. For every pair of distinct sets $X, X' \in \mathcal{F}$, connect a^i_X and $a^i_{X'}$ with a path of length $2r_0$, thus making the set A^i into a $2r_0$-subdivided clique.
- Add a vertex b^i and connect it to every vertex of A^i using a path of length $2r_0$.
- Add a pendant path of length r_0 with one endpoint at b^i. Let the second endpoint of this path be denoted by c^i.

Next, for every $e \in U$ do the following:

- Create a vertex u_e and connect it to every vertex a^i_X such that $i \in [k]$, $X \in \mathcal{F}$ and $e \in X$ using a path of length $2r_0$.
- Add a pendant path of length r_0 with one endpoint at u_e. Let the second endpoint of this path be denoted by v_e.

This concludes the construction. It is easy to see that $G \in \mathcal{H}_{r_0}$, since G consists of the named vertices connected by paths of length r_0 or $2r_0$. It remains to show that instance (G, k) of $3r_0$-Dominating Set is equivalent to the input instance $(U, \mathcal{F}, k)$ of Set Cover.

Claim 6.5. *If instance $(U, \mathcal{F}, k)$ of* Set Cover *has a solution, then so does instance (G, k) of $3r_0$-*Dominating Set.

Proof. Let $\mathcal{G} = \{X_1, X_2, \ldots, X_k\}$ be an arbitrary enumeration of a solution $\mathcal{G}$ to $(U, \mathcal{F}, k)$. Let $D = \{a^i_{X_i} \,:\, i \in [k]\}$. We claim that set D $3r_0$-dominates the graph G. Observe that, by the construction, every vertex of G is at distance at most r_0 from some vertex of $R := \{b^i \,:\, i \in [k]\} \cup \{a^i_X \,:\, i \in [k],\, X \in \mathcal{F}\} \cup \{u_e \,:\, e \in U\}$. Therefore, it suffices to prove that every vertex of R is at distance at most $2r_0$ from a vertex belonging to D.

Firstly, every vertex b^i for $i \in [k]$ is at distance $2r_0$ from $a^i_{X_i}$. Secondly, the same holds also for every vertex $a^i_{X'}$ for every $X' \in \mathcal{F}$, $X' \neq X_i$. Finally, each vertex u_e is at distance $2r_0$ from vertex $a^i_{X_i}$ for any X_i such that $e \in X_i$; since $U = \bigcup \mathcal{G}$, such an index i always exists. By considering all the cases, we conclude that D is indeed a $3r_0$-dominating set in G. ⌟

Claim 6.6. *If instance* (G, k) *of* $3r_0$-Dominating Set *has a solution, then so does instance* $(U, \mathcal{F}, k)$ *of* Set Cover.

Proof. Let D be a solution to (G, k). For every $i \in [k]$, let C^i be the set of vertices at distance at most $3r_0$ from c^i; observe that C^i comprises c^i, b^i, all the vertices of A^i, and all vertices lying on the paths connecting b^i with vertices of $\{c^i\} \cup A^i$. As c^i is $3r_0$-dominated by D, every set C^i has a nonempty intersection with D. As sets C^i are pairwise disjoint and $|D| \leqslant k$, we infer that $|D| = k$, $D \subseteq \bigcup_{i \in [k]} C_i$ and every set C^i contains exactly one vertex of D. Define $\mathcal{G} = \{X_1, X_2, \ldots, X_k\}$ as follows: if $C^i \cap D \subseteq A^i$ then let X_i be such that $C^i \cap D = \{a^i_{X_i}\}$, and otherwise set X_i to be an arbitrary set from $\mathcal{F}$. We claim that $\mathcal{G}$ constructed in this manner is a solution to $(U, \mathcal{F}, k)$.

Take any $e \in U$ and consider the vertex v_e. This vertex has to be dominated by D, however the only vertices of $\bigcup_{i \in [k]} C_i$ that are at distance at most $3r_0$ from v_e are vertices of the form a^i_X for $i \in [k]$ and $X \in \mathcal{F}$ such that $e \in X$. We infer that at least one of these vertices must belong to D, so there exists an index i with the following property: set X_i is chosen so that $C^i \cap D = \{a^i_{X_i}\}$ and moreover $e \in X_i$. Since e was chosen arbitrarily, we conclude that $U \subseteq \bigcup \mathcal{G}$. ⌟

Claims 6.5 and 6.6 verify the correctness of the reduction, and thus the proof is concluded. □