Algorithmica manuscript No.
(will be inserted by the editor)

On the Parameterized Complexity of Reconfiguration of Connected Dominating Sets

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Received: date / Accepted: date

Abstract In a reconfiguration version of a decision problem $\mathcal{Q}$ the input is an instance of $\mathcal{Q}$ and two feasible solutions S and T. The objective is to determine whether there exists a step-by-step transformation between S and T such that all intermediate steps also constitute feasible solutions. In this work, we study the parameterized complexity of the Connected Dominating Set Reconfiguration problem (CDS-R). It was shown in previous work that the Dominating Set Reconfiguration problem (DS-R) parameterized by k, the maximum allowed size of a dominating set in a reconfiguration sequence, is fixed-parameter tractable on all graphs that exclude a biclique $K_{d,d}$ as a subgraph, for some constant $d \geq 1$. We show that the additional connectivity constraint makes the problem much harder, namely, that CDS-R is $\mathsf{W}[1]$-hard parameterized by $k + \ell$, the maximum allowed size of a dominating set plus the length of the reconfiguration sequence, already on 5-degenerate graphs. On

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A preliminary version of this paper was accepted for publication at the 15th International Symposium on Parameterized and Exact Computation, IPEC 2020, December 14-18, 2020, Hong Kong, China.
The second author is supported by URB project "A theory of change through the lens of reconfiguration".

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the positive side, we show that CDS-R parameterized by k is fixed-parameter tractable, and in fact admits a polynomial kernel on planar graphs.

Keywords reconfiguration · parameterized complexity · connected dominating set · graph structure theory

1 Introduction

In a decision problem $\mathcal{Q}$, we are usually asked to determine the existence of a feasible solution for an instance $\mathcal{I}$ of $\mathcal{Q}$. In a *reconfiguration version* of $\mathcal{Q}$, we are instead given a source feasible solution S and a target feasible solution T and we are asked to determine whether it is possible to transform S into T by a sequence of step-by-step transformations such that after each intermediate step we also maintain feasible solutions. Formally, we consider a graph, called the *reconfiguration graph*, that has one vertex for each feasible solution and where two vertices are connected by an edge if we allow the transformation between the two corresponding solutions. We are then asked to determine whether S and T are connected in the reconfiguration graph, or even to compute a shortest path between them. Historically, the study of reconfiguration questions predates the field of computer science, as many classic one-player games can be formulated as such reachability questions [21, 23], e.g., the 15-puzzle and Rubik's cube. More recently, reconfiguration problems have emerged from computational problems in different areas such as graph theory [2, 19, 20], constraint satisfaction [13, 28] and computational geometry [6, 22, 26], and even quantum complexity theory [12]. Reconfiguration problems have been receiving considerable attention in recent literature, we refer the reader to [18, 27, 31] for an extensive overview.

In this work, we consider the Connected Dominating Set Reconfiguration problem (CDS-R) in undirected graphs. A *dominating set* in a graph G is a set $D \subseteq V(G)$ such that every vertex of G lies either in D or is adjacent to a vertex in D. A dominating set D is a *connected dominating set* if the graph induced by D is connected. The Dominating Set problem and its connected variant have many applications, including the modeling of facility location problems, routing problems, and many more [1, 15, 35].

We study CDS-R under the *Token Addition/Removal* model (TAR model). Suppose we are given a connected dominating set D of a graph G, and imagine that a token is placed on each vertex in D. The TAR rule allows either the addition or removal of a single token at a time from D, if this results in a connected dominating set of size at most a given bound $k \geq 1$. A sequence $D_1, \ldots, D_\ell$ of connected dominating sets of a graph G is called a *reconfiguration sequence* between D_1 and D_ℓ under TAR if the change from D_i to D_{i+1} respects the TAR rule, for $1 \leq i < \ell$. The *length* of the reconfiguration sequence is $\ell - 1$.

The (Connected) Dominating Set Reconfiguration problem for TAR gets as input a graph G, two (connected) dominating sets S and T and an integer $k \geq 1$, and the task is to decide whether there exists a reconfiguration sequence between S and T under TAR using at most k tokens.

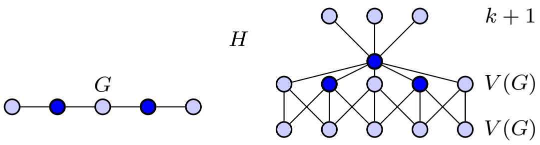


Fig. 1: A graph G with a minimum dominating set of size $k = 2$ marked in dark blue and the graph H obtained in the standard reduction from Dominating Set to Connected Dominating Set. G has a dominating set of size k if and only if H has a connected dominating set of size $k + 1$. If p is equal to the pathwidth of G then the pathwidth of H is bounded by $2p + 1$.

Structural properties of the reconfiguration graph for k-dominating sets were studied in [16, 34]. The Dominating Set Reconfiguration problem was shown to be PSPACE-complete in [17], even on split graphs, bipartite graphs, planar graphs and graphs of bounded bandwidth. Both the pathwidth and the treewidth of a graph are bounded by its bandwidth, hence the Dominating Set Reconfiguration problem is PSPACE-complete on graphs of bounded pathwidth and treewidth. These hardness results motivated the study of the parameterized complexity of the problem. It was shown in [29] that the Dominating Set Reconfiguration problem is $\mathsf{W}[2]$-hard when parameterized by $k + \ell$, where k is the bound on the number of tokens and ℓ is the length of the reconfiguration sequence. However, the problem becomes fixed-parameter tractable (when parameterized by k) on graphs that exclude a fixed complete bipartite graph $K_{d,d}$ as a subgraph, as shown in [25]. Such so-called *biclique-free* classes are very general sparse graph classes, including in particular the planar graphs, which are $K_{3,3}$-free.

In this work we study the complexity of CDS-R. The standard reduction from Dominating Set to Connected Dominating Set shows that CDS-R is also PSPACE-complete, even on graphs of bounded pathwidth (Figure 1). We hence turn our attention to the parameterized complexity of the problem[1]. We first show that the additional connectivity constraint makes the problem much harder, namely, that CDS-R parameterized by $k+\ell$ is $\mathsf{W}[1]$-hard already on 5-degenerate graphs. As 5-degenerate graphs exclude the biclique $K_{6,6}$ as a subgraph, Dominating Set Reconfiguration is fixed-parameter tractable on much more general graph classes than its connected variant. To prove hardness we first introduce an auxiliary problem that we believe is of independent interest. In the Colored Connected Subgraph problem we are given a graph G, an integer k, and a (not necessarily proper) coloring $c\colon V(G) \to C$, for some color set C with $|C| \leq k$. The question is whether G contains a vertex subset H on at most k vertices such that $G[H]$ is connected and H contains at least one vertex of every color in C (i.e., $c(H) = C$). The reconfiguration variant Colored Connected Subgraph Reconfiguration (CCS-R) is defined

[1] We note that the problem is easily shown to be slicewise polynomial parameterized for parameter $k + \ell$ as one can guess each set in the reconfiguration sequence.

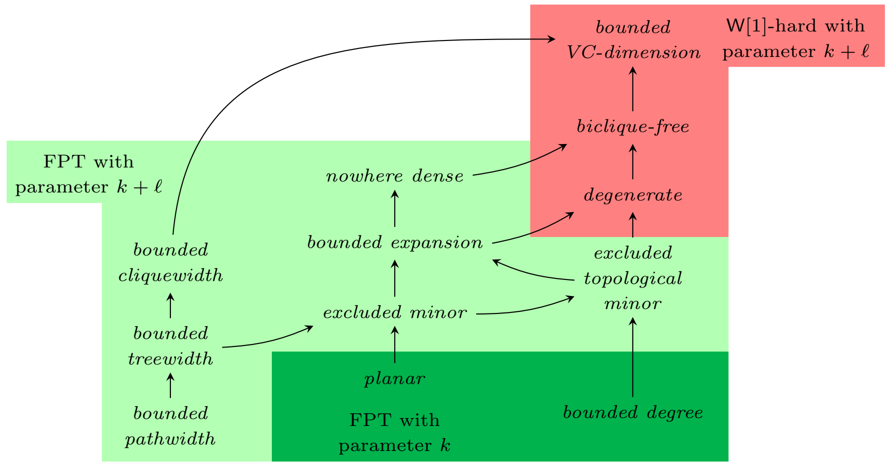


Fig. 2: The map of tractability for Connected Dominating Set Reconfiguration. The classes colored in dark green admit an FPT algorithm with parameter k, the classes colored in light green admit an FPT algorithm with parameter $k+\ell$. On the classes colored in red the problem is W[1]-hard with respect to the parameter $k+\ell$.

as expected. We first prove that CCS-R reduces to CDS-R by a parameter preserving reduction (where $k+\ell$ is the parameter) and the degeneracy of the reduced to graph is at most the degeneracy of the input graph plus one. We then prove that the known W[1]-hard problem Multicolored Clique (see [4] for definitions) reduces to CCS-R on 4-degenerate graphs. The last reduction has the additional property that for an input (G, c, k) of Multicolored Clique the resulting instance of CCS-R admits either a reconfiguration sequence of length $\mathcal{O}(k^3)$, or no reconfiguration sequence at all. Hence, we derive that both CDS-R and CCS-R are W[1]-hard parameterized by $k+\ell$ on 5-degenerate and 4-degenerate graphs, respectively.

The existence of a reconfiguration sequence of length at most ℓ with connected dominating sets of size at most k can be expressed by a first-order formula of length depending only on k and ℓ. It follows from [14] that the problem is fixed-parameter tractable parameterized by $k+\ell$ on every nowhere dense graph class and the same is implied by [3] for every class of bounded cliquewidth. Nowhere dense graph classes are very general classes of uniformly sparse graphs, in particular the class of planar graphs is nowhere dense. Nowhere dense classes are themselves biclique-free, but are not necessarily degenerate. Hence, our hardness result on degenerate graphs essentially settles the question of fixed-parameter tractability for the parameter $k+\ell$ on sparse graph classes. It remains an interesting open problem to find dense graph classes beyond classes of bounded cliquewidth on which the problem is fixed-parameter tractable.

We then turn our attention to the smaller parameter k alone. We show that CDS-R parameterized by k is fixed-parameter tractable on the class of planar graphs. Our approach is as follows. We first compute a small *domination core* for G, a set of vertices that captures exactly the domination properties of G for dominating sets of sizes not larger than k. The notion of a domination core was introduced in the study of the DISTANCE-r DOMINATING SET problem on nowhere dense graph classes [5]. While the classification of interactions with the domination core would suffice to solve DOMINATING SET RECONFIGURATION on nowhere dense classes, additional difficulties arise for the connected variant. In a second step we use planarity to identify large subgraphs that have very simple interactions with the domination core and prove that they can be replaced by constant size gadgets such that the reconfiguration properties of G are preserved.

Observe that CDS-R parameterized by k is trivially fixed-parameter tractable on every class of bounded degree. The existence of a connected dominating set of size k implies that the diameter of G is bounded by $k+2$, which in every bounded degree class implies a bound on the size of the graph depending only on the degree and k. We conjecture that CDS-R is fixed-parameter tractable parameterized by k on every nowhere dense graph class. However, resolving this conjecture remains open for future work (see Figure 2).

The rest of the paper is organized as follows. We give background on graph theory and fix our notation in Section 2. We show hardness of CDS-R on degenerate graphs in Section 3 and show how to handle the planar case in Section 4.

2 Preliminaries

We denote the set of natural numbers by $\mathbb{N}$. For $n \in \mathbb{N}$, we let $[n] = \{1, 2, \ldots, n\}$. We assume that each graph G is finite, simple, and undirected. We let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. An edge between two vertices u and v in a graph is denoted by $\{u, v\}$ or uv. The *open neighborhood* of a vertex v is denoted by $N_G(v) = \{u \mid \{u, v\} \in E(G)\}$ and the *closed neighborhood* by $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v, denoted by $d_G(v)$, is $|N_G(v)|$. For a set of vertices $S \subseteq V(G)$, we define $N_G(S) = \{v \notin S \mid \{u, v\} \in E(G), u \in S\}$ and $N_G[S] = N_G(S) \cup S$. The subgraph of G induced by S is denoted by $G[S]$, where $G[S]$ has vertex set S and edge set $\{\{u, v\} \in E(G) \mid u, v \in S\}$. We let $G - S = G[V(G) \setminus S]$. A graph G is d-degenerate if every subgraph $H \subseteq G$ has a vertex of degree at most d. For a set C, we use $K[C]$ to denote the complete graph on vertex set C. For an integer $r \in \mathbb{N}$, an r-independent set in a graph G is a subset $U \subseteq V(G)$ such that for any two distinct vertices $u, v \in U$, the distance between u and v in G is more than r. An independent set in a graph is a 1-independent set. A subset of vertices U in G is called a separator in G if $G - U$ has more than one connected component. For $s, t \in V(G)$, we say U is an (s, t)-separator in G if there is no path from s to t in $G - U$.

3 Hardness on degenerate graphs

In this section we prove that CDS-R and CCS-R are W[1]-hard when parameterized by $k + \ell$ even on 5-degenerate and 4-degenerate graphs, respectively. Towards that, we first give a polynomial-time reduction from the W[1]-hard Multicolored Clique problem to CCS-R on 4-degenerate graphs with the property that for an input (G, c, k) of Multicolored Clique the resulting instance of CCS-R admits either a reconfiguration sequence of length $\mathcal{O}(k^3)$ or no reconfiguration sequence at all. As a result, we conclude that CCS-R is W[1]-hard when parameterized by $k + \ell$ on 4-degenerate graphs. Then, we give a parameter-preserving polynomial-time reduction from CCS-R to CDS-R.

Let us first formally define the CCS and CCS-R problems.

Colored Connected Subgraph (CCS) **Parameter:** k
Input: A graph G, a vertex-coloring $c\colon V(G) \to C$, and $k \in \mathbb{N}$ such that $\lvert C\rvert \leq k$
Question: Is there a vertex subset $S \subseteq V(G)$ of at most k vertices with at least one vertex from every color class such that $G[S]$ is connected?

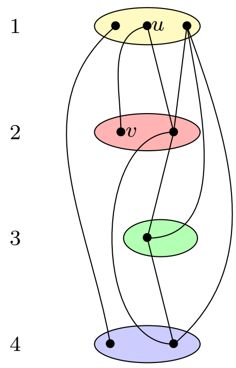
Colored Connected Subgraph Reconf (CCS-R) **Parameter:** k
Input: A graph G, a vertex-coloring $c\colon V(G) \to C$, two sets $Q_s, Q_t \subseteq V(G)$, and $k \in \mathbb{N}$ such that $\lvert C\rvert, \lvert Q_s\rvert, \lvert Q_t\rvert \leq k$, $c(Q_s) = c(Q_t) = C$, and $G[Q_s]$,$G[Q_t]$ are connected
Question: Is there a reconfiguration sequence from Q_s to Q_t?

3.1 Reduction from Multicolored Clique to CCS-R

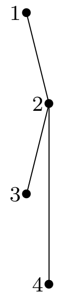
We now present the reduction from Multicolored Clique to CCS-R, which we believe to be of independent interest. We can assume, without loss of generality, that for an input (G, c, k) of Multicolored Clique, G is connected and c is a proper vertex-coloring, i.e., for any two distinct vertices $u, v \in V(G)$ with $c(u) = c(v)$ we have $\{u, v\} \notin E(G)$. Before we proceed let us define a graph operation.

Definition 1 Let G be a graph and let $c\colon V(G) \to \{1, \ldots, k\}$ be a proper vertex coloring of $V(G)$. Let H be a graph on the vertex set $\{1, \ldots, k\}$. We define the graph $G \upharpoonright_c H$ as follows. We remove all edges $\{u, v\} \in E(G)$ such that $c(u) = i$ and $c(v) = j$ and $\{i, j\} \notin E(H)$. We subdivide every remaining edge, i.e., for every remaining edge $\{u, v\}$ we introduce a new vertex s_{uv}, remove the edge $\{u, v\}$ and introduce instead the two edges $\{u, s_{uv}\}$ and $\{v, s_{uv}\}$. We write $W(G \upharpoonright_c H)$ for the set of all subdivision vertices s_{uv} (see Figure 3).

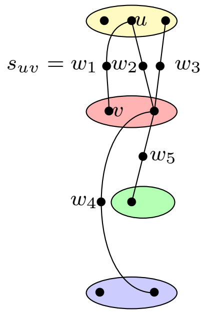
That is, to construct $G \upharpoonright_c H$, we first make a subgraph of G by deleting the edges between different color classes if there are no edges between the "corresponding" vertices in H, and then subdivide the remaining edges. Let (G, c, k) be the input instance of Multicolored Clique, where G is a connected graph and c is a proper k-vertex-coloring of G. We construct an instance $(H, \widehat{c}\colon V(H) \mapsto [k+1], Q_s, Q_t, 2k)$ of CCS-R (Q_s and Q_t are the source and target sets that we describe later). Note that the bound on the sizes of the solutions in the reconfiguration sequence is at most $2k$.



(a) A graph G and a proper coloring $c\colon V(G) \to \{1,\ldots,4\}$



(b) A graph H on the vertex set $\{1,\ldots,4\}$



(c) The graph $G \restriction_c H$. Here, we have $W(G \restriction_c H) = \{w_1,\ldots,w_5\}$

Fig. 3: Construction of $G \restriction_c H$.

We first construct a routing gadget. For $1 \leq i \leq k$, let T^i be the star with vertex set $\{1,\ldots,k\}$ having vertex i as the center. For any $1 \leq i \leq k$ and $1 \leq r \leq 20k$, we let $H^{(i,r)}$ be a copy of the graph $G \restriction_c T^i$. We let $c_{(i,r)}$ be the the partial vertex-coloring of $H^{(i,r)}$ that is naturally inherited from G. For an illustration, consider the input instance (G, c, k) of Multicolored Clique depicted in Figure 3a. Then, T^2 is identical to the graph H in Figure 3b and Figure 3c represents $H^{(2,r)} = G \restriction_c T^2$, for any $1 \leq r \leq 20k$. Now, for $1 \leq i \leq k$ we define a graph H^i as follows. We use $W(H^{(i,r)})$ to denote the set of subdivision vertices in $H^{(i,r)}$. For $1 \leq r < 20k$ and all vertices u, v in $V(H^{(i,r)}) \setminus W(H^{(i,r)})$, we connect the copy of the subdivision vertex s_{uv} in $H^{(i,r)}$ (if it exists) with the copies of the vertices u and v in $H^{(i,r+1)}$ (see Figure 4 for an illustration of a portion of H^1 and Figure 5 for an illustration of a portion of H^2). We use $W(H^i)$ to denote the set of subdivision vertices $\bigcup_{r\in[20k]} W(H^{(i,r)})$.

For each $1 \leq i \leq k$, we use c_i to denote a coloring on $V(H^i)$ that is the union of $c_{(i,1)}, c_{(i,2)}, \ldots, c_{(i,20k)}$ and we color all the copies of the subdivision vertices using a new color $k+1$. In other words, we know that for each $u \in V(H^i)$ we have $u \in V(H^{(i,r)})$, for some $r \in \{1,\ldots,20k\}$. Hence, if $u \in V(H^{(i,r)}) \setminus W(H^{(i,r)})$ then we set $c_i(u) = c_{(i,r)}(u)$. For all $s_{uv} \in W(H^i)$, we set $c_i(s_{uv}) = k+1$.

Now, define a graph R, which is supergraph of $H^1 \cup \ldots \cup H^k$, as follows. For $1 \leq i < k$ and all vertices u and v, we connect the copy of the subdivision

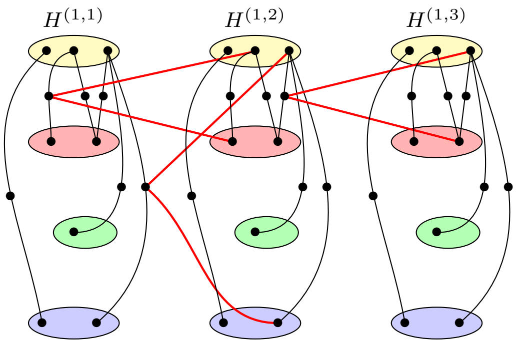


Fig. 4: Construction of H^1 from the instance (G, c, k) depicted in Figure 3a. The red edges are some of the "crossing" edges but not all of them.

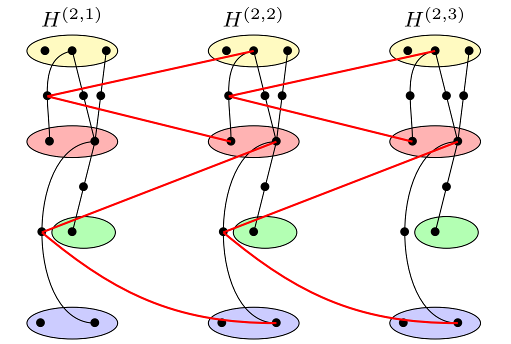


Fig. 5: Construction of H^2 from the instance (G, c, k) depicted in Figure 3a. The red edges are some of the "crossing" edges but not all of them.

vertex s_{uv} in $H^{(i,20k)}$ (if it exists) with the copies of the vertices u and v in $H^{(i+1,1)}$ (see Figure 6 for an illustration).

We additionally introduce two subgraphs H^0 and H^{k+1}. The graph H^0 is obtained by subdividing each edge of a star on vertex set $\{v_1, \ldots, v_k\}$ centered at v_1. Here we use $w_2, \ldots, w_k$ to denote the subdivision vertices. Similarly, the graph H^{k+1} is obtained by subdividing each edge of star on $\{x_1, \ldots, x_k\}$ centered at x_k. Here $y_1, \ldots, y_{k-1}$ denote the subdivision vertices. Let c_0 and c_{k+1} be the colorings on $\{v_1, \ldots, v_k, w_2, \ldots, w_k\}$ and $\{x_1, \ldots, x_k, y_1, \ldots, y_{k-1}\}$, respectively, defined as follows. For all $1 \leq i \leq k$, $c_0(v_i) = i$ and $c_{k+1}(x_i) = i$. For all $2 \leq i \leq k$, $c_0(w_i) = k + 1$ and for all $1 \leq i \leq k - 1$, $c_{k+1}(y_i) = k + 1$. Observe that we may interpret H^0 as $K[\{v_1, \ldots, v_k\}] \restriction_{c_0} T^0$ and H^{k+1} as $K[\{x_1, \ldots, x_k\}] \restriction_{c_{k+1}} T^k$, where T^0 and T^k are two stars on vertex set $\{1, \ldots, k\}$, with $E(T^0) = \{\{1, i\} \colon 2 \leq i \leq k\}$ and $E(T^k) = \{\{k, i\} \colon 1 \leq i \leq k - 1\}$ (as previously defined).

Finally, for each $2 \leq i \leq k$, we connect the "subdivision vertex" w_i (adjacent to v_1 and v_i) to all vertices $v \in V(H^{(1,1)})$ colored 1 or i, i.e., with $c_{(1,1)}(v) \in$

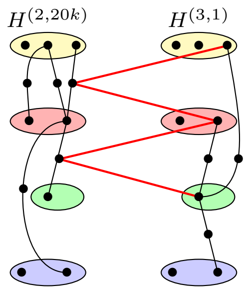


Fig. 6: Illustration of the subgraph of R induced on $V(H^{(2,20k)}) \cup V(H^{3,1})$ constructed from the instance (G, c, k) depicted in Figure 3a. The red edges are some of the "crossing edges".

$\{1, i\}$. For each subdivision vertex $s_{ab} \in W(H^{(k,20k)})$, we connect s_{ab} to x_k and x_i, where $k = c_k(a) = c_{(k,20k)}(a)$ and $i = c_k(b) = c_{(k,20k)}(b)$. Recall that s_{ab} is adjacent to a vertex of color k and a vertex of color i, for some $i < k$. This completes the construction of H (see Figure 7). We define $\widehat{c}\colon V(H) \mapsto [k+1]$ to be the union of $c_0, \ldots, c_{k+1}$. We define the starting configuration Q_s as the set $\{v_1, \ldots, v_k, w_2, \ldots, w_k\}$ and the target configuration Q_t as the set $\{x_1, \ldots, x_k, y_1, \ldots, y_{k-1}\}$.

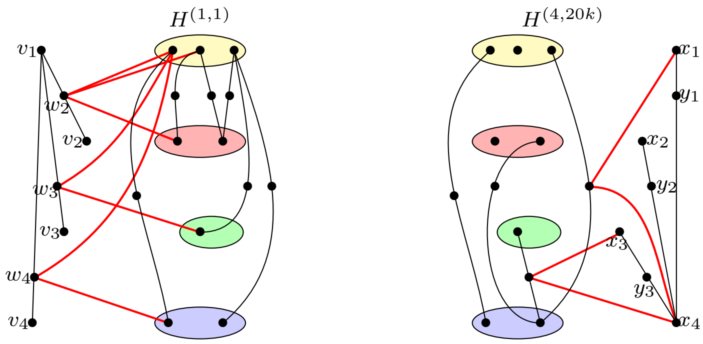


Fig. 7: Illustration of connections between H^0 and R, and H^{k+1} and R from the instance (G, c, k) depicted in Figure 3a. The red edges are some of the "crossing edges" between H^0 and H^1, and H^k and H^{k+1}.

Proposition 1 *The sets Q_s and Q_t are solutions of size $2k-1$ of the* CCS *instance $(H, \widehat{c}, 2k)$.*

We now consider the instance $(H, \widehat{c}, Q_s, Q_t, 2k)$ of the CCS-R problem. Let us give some high-level intuition about the construction before proceeding to formal proofs. Assuming that (G, c, k) is a yes-instance of Multicolored

Clique, we show how to construct a reconfiguration sequence from Q_s to Q_t as follows. Our goal is to shift the connected vertices of Q_s through the subgraphs $H^1, \ldots, H^k$ (in that order) while maintaining connectivity and eventually reaching Q_t. To do so, we use the corresponding vertices of the clique in each $H^{i,j}$ to maintain colorful sets and we use the vertices corresponding to subdivided edges to maintain connectivity. In the reverse direction, we shall show that in any reconfiguration sequence, each part of the constructed graph, i.e., each H^i, will allow us to guarantee that there exists a vertex colored i that is connected to vertices of every other color (while maintaining the choice of vertices along the way).

Before we analyze the reconfiguration properties of H, let us first verify that H is 4-degenerate.

Lemma 1 *The graph H is 4-degenerate.*

Proof We iteratively remove minimum degree vertices and show that we can always remove a vertex of degree at most 4 in each step.

- Every subdivision vertex $w \in W(H^i)$ for $1 \leq i \leq k$ has degree at most 4; it has 4 neighbors in $V(H^i) \cup V(H^{i+1})$.
- After removal of all subdivision vertices the degree of the remaining vertices of each H^i is at most one. That is, a vertex in $H^{(1,1)}$ may have a neighbor in $\{w_2, \ldots, w_k\}$.
- After the removal of $V(H^1) \cup \ldots V(H^k)$, the degree of all vertices except v_1 and x_k is at most 2.
- Finally we remove v_1 and x_k.

This completes the proof. □

Lemma 2 *Let T_1, T_2 be two trees on vertex set $\{1, \ldots, k\}$ and let $f_1, \ldots f_{k-1}$ be an ordering of the edges in T_2. Then, in polynomial time, we can find an ordering $e_1, \ldots, e_{k-1}$ of the edges in T_1 such that the following holds. In the sequence of graphs $T'_0, T'_1, \ldots, T'_{k-1}$ on vertex set $\{1, \ldots, k\}$, where for each $0 \leq i \leq k-2$, $T'_{i+1} = T'_i + f_i - e_i$ and $T'_0 = T_1$, we have that T'_i is a tree, for all $i \in [k-1]$, and $T'_{k-1} = T_2$.*

Proof We proceed by induction on $\ell = |E(T_1) \setminus E(T_2)|$. In the base case, we have $\ell = 0$ and $E(T_1) = E(T_2)$. In this case $f_1, \ldots f_{k-1}$ is also the required ordering of the edges in T_1 (note that the sequence of graphs consists of only $T_1 = T_2$ in this case).

Now consider the induction step, $\ell > 1$. Let j be the first index in $\{1, \ldots, k-1\}$ such that $f_j \notin E(T_1)$. We add f_j to T_1 and this creates a cycle in T_1. Hence, there exists an edge $e_j \in E(T_1) \setminus E(T_2)$ whose removal results in a tree. That is, $T'_1 = T_1 + f_j - e_j$ is a tree. Notice that $|E(T'_1) \setminus E(T_2)| = \ell - 1$. By the induction hypothesis, there is a sequence $g_1, \ldots, g_{k-1}$ of edges in $E(T'_1)$ such that for the sequence of graphs $T'_1 = T''_0, T''_1, \ldots, T''_{k-1}$ on vertex set $\{1, \ldots, k\}$, we have $T''_{i+1} = T''_i + f_i - g_i$, each T''_i is a tree, and $T_2 = T''_{k-1}$, $0 \leq i < k$. Since j is the first index in $\{1, \ldots, k-1\}$ such that $f_j \notin E(T_1)$, $T'_1 = T_1 + f_j - e_j$, and $T_0'', T_1'', \ldots, T_{k-1}''$ are trees, we have that $g_i = f_i$ for all $i < j$. Notice that $f_j \in E(T_1')$ and $E(T_1) = (E(T_1') \setminus \{f_j\}) \cup \{e_j\}$.

We claim that $e_1, \ldots, e_{j-1}, e_j, e_{j+1}, \ldots, e_{k-1}$, where $e_i = g_i$ for all $i < j$, is the required sequence of edges in T_1. Let $T_0', T_1', \ldots, T_{k-1}'$ be the sequence where, for each $0 \le i \le k-2$, $T_{i+1}' = T_i' + f_i - e_i$ and $T_0' = T_1$. Since $g_i = f_i = e_i$ for all $i < j$, we have that $T_1 = T_0' = T_1' = \ldots = T_{j-1}'$. Moreover, $T_j' = T_1 + \{f_1, \ldots, f_j\} - \{e_1, \ldots, e_j\} = T_1 + \{f_1, \ldots, f_j\} - \{g_1, \ldots, g_j\} = T_j''$ because $E(T_1) = (E(T_1') \setminus \{f_j\}) \cup \{e_j\}$ and $e_i = g_i$ for all $i < j$. Then, the sequence $T_j', \ldots, T_{k-1}'$ is the same as the sequence $T_j'', \ldots, T_{k-1}''$. Therefore, the sequence $e_1, \ldots, e_{j-1}, e_j, e_{j+1}, \ldots, e_{k-1}$ of edges in T_1 satisfies the conditions of the lemma. □

Lemma 3 *If there exists a k-colored clique in G then there is reconfiguration sequence of length $\mathcal{O}(k^3)$ from Q_s to Q_t in $(H, \widehat{c}, 2k)$.*

Proof We aim to shift the connected vertices of Q_s through the subgraphs $H^1, \ldots, H^k$ (in that order) to maintain connectivity and eventually shift to Q_t. For each $u_i \in V(G)$, $1 \le j \le k$ and $1 \le r \le 20k$, we use $u_i^{(j,r)}$ to denote the copy of u_i in $H^{(j,r)}$.

Let $C = \{u_1, \ldots, u_k\}$ be a k-colored clique in G such that $c(u_i) = i$, for all $1 \le i \le k$. To prove the lemma, we need to define a reconfiguration sequence starting from Q_s and ending at Q_t such that the cardinality of any solution in the sequence is at most $2k$. First we define k "colored" trees $\widehat{T}_1, \ldots, \widehat{T}_k$ each on $2k-1$ vertices, and then prove that there are reconfiguration sequences from Q_s to $V(\widehat{T}_1)$, $V(\widehat{T}_i)$ to $V(\widehat{T}_{i+1})$ for all $1 \le i < k$, and $V(\widehat{T}_k)$ to Q_t.

We start by defining $\widehat{T}_1, \ldots, \widehat{T}_k$. For each $1 \le i \le k$, $C_i = \{u_1^{(i,1)}, \ldots, u_k^{(i,1)}\}$ and $S_i = \{z \in V(H^{(i,1)}) \colon |N_{H^{(i,1)}}(z) \cap C_i| = 2\}$. That is, for each $1 \le j \le k$ and $j \ne i$, $s_{u_i^{(i,1)} u_j^{(i,1)}} \in S_i$ (the subdivision vertex on the edge $u_i^{(i,1)} u_j^{(i,1)}$ is in S_i), and $|S_i| = k-1$. In other words, C_i contains the copies of the vertices of the clique C in $H^{(i,1)}$ and S_i contains subdivision vertices corresponding to $k-1$ edges in the clique incident on the ith colored vertex of the clique, such that $H[C_i \cup S_i]$ is a tree. Now, define $\widehat{T}_i = H[C_i \cup S_i]$. It is easy to verify that $\widehat{c}(C_i \cup S_i) = \{1, \ldots, k+1\}$ and hence $C_i \cup S_i = V(\widehat{T}_i)$ is a solution to the CCS instance $(H, \widehat{c}, 2k)$. Let $T_s = H[Q_s]$ and $T_t = H[Q_t]$. Note that T_s and T_t are trees on $2k-1$ vertices each.

Case 1: Reconfiguration from Q_s to $V(\widehat{T}_1)$. Informally, we move to $\widehat{T}_1$ by adding a token on $u_i^{(1,1)}$ and then removing a token from v_i for i in the order $2, \ldots, k, 1$ (for a total of $2k$ token additions/removals). Finally, we move the tokens from $\{w_2, \ldots, w_{k-1}\}$ to S_1 in $2(k-1)$ steps. The length of the reconfiguration sequence is $2k + 2(k-1) = 4k-2$.

Formally, we define $Z_0 = Q_s$ and for each $1 \le j \le k-1$, $Z_{2j-1} = Z_{2j-2} \cup \{u_{j+1}^{(1,1)}\}$ and $Z_{2j} = Z_{2j-1} \setminus \{v_{j+1}\}$. That is, for each $1 \le j \le k-1$,

$$Z_{2j-1} = \{u_2^{(1,1)}, \ldots, u_{j+1}^{(1,1)}\} \cup \{v_{j+1} \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}, \text{ and}$$
$$Z_{2j} = \{u_2^{(1,1)}, \ldots, u_{j+1}^{(1,1)}\} \cup \{v_{j+2} \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}.$$

Next, we define Z_{2k-1} and Z_{2k} as

$$Z_{2k-1} = \{u_2^{(1,1)}, \ldots, u_k^{(1,1)}, u_1^{(1,1)}\} \cup \{v_1\} \cup \{w_1, \ldots, w_{k-1}\}, \text{ and}$$
$$Z_{2k} = \{u_2^{(1,1)}, \ldots, u_k^{(1,1)}, u_1^{(1,1)}\} \cup \{w_1, \ldots, w_{k-1}\}.$$

In other words, the first five sets in the reconfiguration sequence look as follows:

$$Z_0 = \{v_2, \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}$$
$$Z_1 = \{u_2^{(1,1)}\} \cup \{v_2, \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}$$
$$Z_2 = \{u_2^{(1,1)}\} \cup \{v_3, \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}$$
$$Z_3 = \{u_2^{(1,1)}, u_3^{(1,1)}\} \cup \{v_3, \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}$$
$$Z_4 = \{u_2^{(1,1)}, u_3^{(1,1)}\} \cup \{v_4, \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}$$
$$Z_5 = \{u_2^{(1,1)}, u_3^{(1,1)}, u_4^{(1,1)}\} \cup \{v_4, \ldots, v_k, v_1\} \cup \{w_1, \ldots, w_{k-1}\}.$$

It is easy to verify that $Z_1, \ldots Z_{2k}$ are solutions to the CCS instance $(H, \widehat{c}, 2k)$. Thus, we now have a reconfiguration sequence $Z_0, Z_1, \ldots, Z_{2k}$, where $Z_0 = Q_s$.

Next, we explain how to get a reconfiguration sequence from Z_{2k} to $V(\widehat{T}_1)$. Recall that $Z_{2k} = C_1 \cup \{w_1, \ldots, w_{k-1}\}$ and $V(\widehat{T}_1) = C_1 \cup S_1$. Let $s_j = s_{u_1^{(1,1)} u_j^{(1,1)}}$, for all $2 \leq j \leq k$. Notice that $S_1 = \{s_2, \ldots, s_k\}$. To obtain a reconfiguration sequence from Z_{2k} to $V(\widehat{T}_1)$, we add s_j and then remove w_j for j in the order $2, \ldots, k$. Since w_j and s_j connect the same two vertices from C_1, this reconfiguration sequence will maintain connectivity. Moreover, it is easy to verify that each set in the reconfiguration sequence uses all the colors $\{1, \ldots, k+1\}$. Therefore, there exists a reconfiguration sequence of length $4k-2$ from Q_s to $V(\widehat{T}_1)$.

Case 2: Reconfiguration from $V(\widehat{T}_i)$ to $V(\widehat{T}_{i+1})$. First we define $20k$ trees $P_1, \ldots P_{20k}$, each on $2k-1$ vertices such that for all $1 \leq r \leq 20k$, (i) $V(P_r) \subseteq V(H^{(i,r)})$, and (ii) $\widehat{T}_i = P_1$. Then we give a reconfiguration sequence from $V(P_r)$ to $V(P_{r+1})$ for all $r \in [20k-1]$ and a reconfiguration sequence from $V(P_{20k})$ to $V(\widehat{T}_{i+1})$.

Recall that $C = \{u_1, \ldots, u_k\}$ is a k-colored clique in G such that $c(u_i) = i$ for all $1 \leq i \leq k$. For each $1 \leq r \leq 20k$, let $C_i^r = \{u_1^{(i,r)}, \ldots, u_k^{(i,r)}\}$ and $S_i^r = \{z \in V(H^{(i,r)}) \colon N_{H^{(i,r)}}(z) \cap C_i^r = 2\}$. That is, for each $1 \leq j \leq k$ and $j \neq i$, $s_{u_i^{(i,r)} u_j^{(i,r)}} \in S_i^r$ (i.e, the subdivision vertex on the edge $u_i^{(i,r)} u_j^{(i,r)}$ is in S_i^r) and $|S_i^r| = k-1$. Let $P_r = H[C_i^r \cup S_i^r]$. Notice that for all $r \in [20k]$, P_r is a tree on $2k-1$ vertices. Moreover, for each $1 \leq r \leq 20k$, $V(P_r)$ is a solution to the CCS instance $(H, \widehat{c}, 2k)$.

Case 2(a): Reconfiguration from $V(P_r)$ to $V(P_{r+1})$. By arguments similar to those given for Case 1, one can prove that there is a reconfiguration sequence of length $4k-2$ from $V(P_r)$ to $V(P_{r+1})$, for all $1 \leq r < 20k$. For completeness we give the details here. Fix an integer $1 \leq r < 20k$. Let $s_j = s_{u_i^{i,r} u_j^{(i,r)}}$ and

$s'_j = s_{u_i^{(i,r+1)} u_j^{(i,r+1)}}$ for all $j \in \{1,\ldots,k\} \setminus \{i\}$. Notice that $S_i^r = \{s_j : j \in \{1,\ldots,k\} \setminus \{i\}\}$ and $S_i^{r+1} = \{s'_j : j \in \{1,\ldots,k\} \setminus \{i\}\}$. Now we define $Z_0 = V(P_r) = C_i^r \cup S_i^r$ and for each $1 \le j \le i-1$, $Z_{2j-1} = Z_{2j-2} \cup \{u_j^{(i,r+1)}\}$ and $Z_{2j} = Z_{2j-1} \setminus \{u_j^{(i,r)}\}$. That is, for each $1 \le j \le i-1$,

$$Z_{2j-1} = \{u_1^{(i,r+1)},\ldots,u_j^{(i,r+1)}\} \cup \{u_j^{(i,r)} \ldots, u_k^{(i,r)}\} \cup S_i^r, \text{ and}$$
$$Z_{2j} = \{u_1^{(i,r+1)},\ldots,u_j^{(i,r+1)}\} \cup \{u_{j+1}^{(i,r)} \ldots, u_k^{(i,r)}\} \cup S_i^r.$$

For each $i \le j \le k-1$, $Z_{2j-1} = Z_{2j-2} \cup \{u_{j+1}^{(i,r+1)}\}$ and $Z_{2j} = Z_{2j-1} \setminus \{u_{j+1}^{(i,r)}\}$. That is, for each $i \le j \le k-1$,

$$Z_{2j-1} = \{u_1^{(i,r+1)},\ldots,u_{i-1}^{(i,r+1)}, u_{i+1}^{(i,r+1)},\ldots,u_{j+1}^{(i,r+1)}\} \cup \{u_{j+1}^{(i,r)} \ldots, u_k^{(i,r)}, u_i^{(i,r)}\} \cup S_i^r$$

and

$$Z_{2j} = \{u_1^{(i,r+1)},\ldots,u_{i-1}^{(i,r+1)}, u_{i+1}^{(i,r+1)},\ldots,u_{j+1}^{(i,r+1)}\} \cup \{u_{j+2}^{(i,r)} \ldots, u_k^{(i,r)}, u_i^{(i,r)}\} \cup S_i^r.$$

Next, we define Z_{2k-1} and Z_{2k} as

$$Z_{2k-1} = \{u_1^{(i,r+1)},\ldots,u_k^{(i,r+1)}\} \cup \{u_i^{(i,r)}\} \cup S_i^r, \text{ and}$$
$$Z_{2k} = \{u_1^{(i,r+1)},\ldots,u_k^{(i,r+1)}\} \cup S_i^r.$$

Next, for each $1 \le j \le k-1$, let $Z_{2k+2j-1} = Z_{2k+2j-2} \cup \{s'_j\}$ and $Z_{2k+2j} = Z_{2k+2j-1} \setminus \{s_j\}$. It is easy to verify that $Z_1, \ldots Z_{4k-2}$ are solutions to the CCS instance $(H, \widehat{c}, 2k)$ and $Z_0, \ldots, Z_{4k-2}$ is a reconfiguration sequence where $Z_0 = V(P_r)$ and $Z_{4k-2} = V(P_{r+1})$.

Case 2(b): Reconfiguration from $V(P_{20k})$ to $V(\widehat{T}_{i+1})$. Next, we explain how to get a reconfiguration sequence from $V(P_{20k})$ to $V(\widehat{T}_{i+1})$ using Lemma 2. Recall that we have

$$C_i^{20k} = \{u_1^{(i,20k)},\ldots,u_k^{(i,20k)}\} \text{ and}$$
$$S_i^{20k} = \{z \in V(H^{(i,20k)}) : |N_{H^{(i,20k)}}(z) \cap C_i^{20k}| = 2\}.$$

Let $C_{i+1} = \{u_1^{(i+1,1}, \ldots, u_k^{(i+1,1)}\}$ and $S_{i+1} = \{z \in V(H^{(i+1,1)}) :\ N_{H^{(i+1,1)}}(z) \cap C_{i+1} = 2\}$. For ease of presentation, let $s_j = s_{u_i^{(i,20k)} u_j^{(i,20k)}}$ for all $j \in \{1,\ldots,k\} \setminus \{i\}$. Also, let $s'_j = s_{u_i^{(i+1,1)} u_j^{(i+1,1)}}$ for all $j \in \{1,\ldots,k\} \setminus \{i+1\}$. That is, $S_i^{20k} = \{s_j : j \in \{1,\ldots,k\} \setminus \{i\}\}$ and $S_{i+1} = \{s'_j : j \in \{1,\ldots,k\} \setminus \{i+1\}\}$. Notice that $V(P_{20k}) = C_i^{20k} \cup S_i^{20k}$ and $V(\widehat{T}_{i+1}) = C_{i+1} \cup S_{i+1}$.

Towards proving the required reconfiguration sequence, we give a reconfiguration sequence from $C_i^{20k} \cup S_i^{20k}$ to $C_{i+1} \cup S_i^{20k}$ and then from $C_{i+1} \cup S_i^{20k}$ to $C_{i+1} \cup S_{i+1}$. The reconfiguration sequence from $C_i^{20k} \cup S_i^{20k}$ to $C_{i+1} \cup S_i^{20k}$

is similar to the one in Case 1. That is, we add $u_j^{(i+1,1)}$ and delete $u_j^{(i,20k)}$ for j in the order $1,\ldots,i-1,i+1,\ldots,k,i$. This gives a reconfiguration sequence from $C_i^{20k} \cup S_i^{20k}$ to $Z = C_{i+1} \cup S_i^{20k}$ of length $2k$.

Next we explain how to get a reconfiguration sequence from $Z = C_{i+1} \cup S_i^{20k}$ to $C_{i+1} \cup S_{i+1}$. Notice that $H[Z]$ and $\widehat{T}_{i+1} = H[C_{i+1} \cup S_{i+1}]$ are trees. Recall that T^i is the star on $\{1,\ldots,k\}$ with vertex i being the center, and T^{i+1} is is the star on $\{1,\ldots,k\}$ with vertex i being the center. Also, c_j is a coloring on H^j which is inherited from the coloring c of G. That is, $c_{i+1}(u_j^{(i+1,1)}) = j$ for all $1 \leq j \leq k$. Then, $H[Z] = K[C_{i+1}] \restriction_{c_{i+1}} T^i$ and $\widehat{T}_{i+1} = H[C_{i+1} \cup S_{i+1}] = K[C_{i+1}] \restriction_{c_{i+1}} T^{i+1}$.

Let $e_1^{i+1},\ldots,e_{k-1}^{i+1}$ be an arbitrary ordering of the the edges in T^{i+1}. By Lemma 2, we have a sequence $e_1^i,\ldots,e_{k-1}^i$ of edges in T^i such that for the sequence $T_0^i, T_1^i, \ldots, T_{k-1}^i$ on vertex set $\{1,\ldots,k\}$, where for each $0 \leq j \leq k-2$, $T_{j+1}^i = T_j^i + e_j^{i+1} - e_j^i$ and $T_0^i = T^i$, the following holds.

(i) T_j^i is a tree for all $0 \leq j \leq k-1$, and
(ii) $T_{k-1}^i = T^{i+1}$.

This implies that, from the sequences $e_1^i,\ldots,e_{k-1}^i$ and $e_1^{i+1},\ldots,e_{k-1}^{i+1}$, we get a sequence $f_1,\ldots,f'_{k-1}$ on S_i^{20k} and a sequence $f'_1,\ldots,f'_{k-1}$ on S_{i+1} such that the for the sequence $L_0,\ldots,L_{2(k-1)}$, where $L_0 = C_{i+1} \cup \{f_1,\ldots,f_{k-1}\}$ and for all $1 \leq j \leq k-1$ $L_{2j-1} = (L_{2j-2} \cup \{f'_i\})$, $L_{2j} = L_{2j-1} \setminus \{f_i\}$ the following holds.

(1) $H[L_i]$ is connected for all $0 \leq i \leq k-1$, and
(2) $L_{k-1} = S_{i+1} \cup C_{i+1}$.

Here, conditions (1) and (2) follow from conditions (i) and (ii), respectively. Moreover, $\widehat{c}(L_i) = [k+1]$ for all $0 \leq i \leq 2(k-1)$ and $L_0 = Z$. Thus, $L_0,\ldots,L_{2(k-1)}$ is a valid reconfiguration sequence from Z to $V(\widehat{T}_{i+1})$. Note that the ordering on the edges implies an ordering by which we can move the subdivision vertices from S_i to S_{i+1} without violating connectivity. This implies that there is a reconfiguration sequence from $V(P_{20k})$ to $V(\widehat{T}_{i+1})$, of length $4k-2$. Therefore, we have a reconfiguration sequence from $V(\widehat{T}_i)$ to $V(\widehat{T}_{i+1})$ of length $\mathcal{O}(k^2)$.

Case 3: Reconfiguration from $V(\widehat{T}_k)$ to $V(T_t)$. The arguments for this case are similar to those given in Case 1, we therefore omit the details. By summing up the lengths of reconfiguration sequences, we get that if (G,c,k) is a yes-instance of Multicolored Clique then there is a reconfiguration sequence from Q_s to Q_t, of length $\mathcal{O}(k^3)$. □

Lemma 4 *If there is a reconfiguration sequence from Q_s to Q_t then there is a k-colored clique in G.*

Proof For each $1 \leq i \leq k+1$, let R_i be the set of vertices colored by the color i. That is, $R_i = \widehat{c}^{-1}(i)$. First, we prove some auxiliary claims. The proofs of the following two claims follow from the construction of H and the definition of $\widehat{c}$.

Claim 1 *(i) $R_1 \cup \ldots \cup R_k$ is an independent set in H, and (ii) every vertex in R_{k+1} is connected to vertices of at most two distinct colors.*

Claim 2 *Let $v, w \in V(H) \setminus (V(H^0) \cup V(H^{k+1}))$ be two distinct vertices such that $\widehat{c}(v) = \widehat{c}(w)$ and $\widehat{c}(v) \in \{1, \ldots, k\}$. If v and w have a common neighbor in $V(H) \setminus V(H^0)$, then v and w are copies of same vertex $z \in V(G)$.*

Claim 3 *Let $Y \subseteq V(H)$ be a vertex subset such that $\widehat{c}(Y) = \{1, \ldots, k+1\}$ and $H[Y]$ is connected. Then, $|Y| \geq 2k-1$.*

Proof Let $B = Y \setminus \widehat{c}^{-1}(k+1) = Y \cap (R_1 \cup \ldots \cup R_k)$. Since $\widehat{c}(Y) = \{1, \ldots, k+1\}$, $|B| \geq k$ and by Claim 1(i), B is an independent set in H. By Claim 1(ii), each vertex in R_{i+1} is connected to vertices of at most two distinct colors. Thus, since $H[Y]$ is connected, the claim follows. ☐

Suppose $(H, \widehat{c}, Q_s, Q_t, 2k)$ is a yes-instance of CCS-R. Then, there is a reconfiguration sequence $D_1, \ldots, D_\ell$ for $\ell \in \mathbb{N}$, where $D_1 = Q_s$ and $D_\ell = Q_t$. Without loss of generality, we assume that the sequence $D_1, \ldots, D_\ell$ is a minimal reconfiguration sequence. Then, by Claim 3, for each $i \in [\ell]$, $2k-1 \leq |D_i| \leq 2k$.

Moreover, since $|D_1| = |D_\ell| = 2k-1$, we have that for each even i, D_i is obtained from D_{i-1} by a token addition, and for each odd i, D_i is obtained from D_{i-1} by a token removal. This also implies that for each even i, $|D_i| = 2k$, for each odd i, $|D_i| = 2k-1$, and ℓ is odd.

Claim 4 *Let $i \in [\ell]$ and $|D_i| = 2k-1$. Then, for all $1 \leq j \leq k$, $|D_i \cap R_j| = 1$, and $|D_i \cap R_{k+1}| = k-1$. Moreover, each vertex in $D_i \cap R_{k+1}$ will be adjacent to exactly two vertices in $H[D_i]$ and these vertices will be of different colors from $\{1, \ldots, k\}$.*

Proof By Claim 1, $R_1 \cup \ldots \cup R_k$ is independent and every vertex of R_{k+1} is adjacent to vertices of at most two different color classes. Hence, we need at least $k-1$ vertices from R_{k+1} that make the connections between the vertices of D_i colored with $\{1, \ldots, k\}$. The above statement along with the assumption $|D_i| = 2k-1$ imply the claim. ☐

Claim 5 *Let $i \in \{2, \ldots \ell-1\}$. Let $v \in D_i$ and $w \in D_{i+1}$ such that $v, w \notin V(H^0) \cup V(H^{k+1})$, at most one vertex in $\{v, w\}$ is in $V(H^{(1,1)})$, and $\widehat{c}(v) = \widehat{c}(w) \in \{1, \ldots, k\}$. Then, v and w are copies of the same vertex in G. Moreover, $v, w \in V(H^j) \cup V(H^{j+1})$ for some $j \in [k-1]$.*

Proof Suppose v and w are not copies of the same vertex $z \in V(G)$. We know that $|D_i| = 2k-1$ or $|D_i| = 2k$.

Case 1: $|D_i| = 2k-1$. Since D_i is a solution, D_i induces a connected subgraph in H. By Claim 4, $|D_i \cap R_j| = 1$ for all $j \in \{1, \ldots, k\}$ and $|D_i \cap R_{k+1}| = k-1$. Also, by Claim 1, (i) $R_1 \cup \ldots \cup R_k$ is an independent set in H, and (ii) every vertex in R_{k+1} is connected to vertices of at most two distinct colors. Statements (i) and (ii), and the fact that $|D_i| = 2k-1$ imply that (iii) $H[D_i]$ is a tree and each vertex in $D_i \cap R_{k+1}$ is incident to exactly two vertices in D_i. Since

$|D_{i+1}| = |D_i| + 1$, in reconfiguration step $i+1$, we add a vertex to obtain D_{i+1}. We know that $v \in D_i$. Since, for any color $q \in [k]$, there is exactly one vertex in D_i of color q (i.e., $|D_i \cap R_q| = 1$), we have that $D_{i+1} = D_i \cup \{w\}$. Moreover, in step $i+2$, the vertex removed from D_{i+1} will be from $\{v, w\}$ and that vertex will be v (because of the minimality assumption of the length of the reconfiguration sequence). That is, $D_{i+2} = (D_i \cup \{w\}) \setminus \{v\}$. Notice that $|D_i| = |D_{i+2}| = 2k-1$. Let b a vertex in D_{i+2} which is adjacent to w in $H[D_{i+2}]$. Since $R_{k+1} \cap D_i = R_{k+1} \cap D_{i+2}$ and $|D_i| = |D_{i+2}| = 2k-1$, by Claim 1, the neighbors of b in $H[D_i]$ and $H[D_{i+2}]$ are of the same color. This implies that b is adjacent to v in $H[D_i]$. Thus, we proved that $\{b, w\}, \{b, v\} \in E(H)$. If $b \in V(H^0)$, then $v, w \in V(H^{(1,1)})$ which is a contradiction to the assumption. Otherwise, by Claim 2, we conclude that v and w are copies of same vertex.

Case 2: $|D_i| = 2k$. In this case D_{i+1} is obtained by removing a vertex from D_i. Moreover, $i \geq 3$, because we have two vertices in D_i from $V(H) \setminus D_1$. Since $|D_{i+1}| = 2k-1$, because of Claim 4, D_{i+1} is obtained by removing the vertex v from D_i. That is, $D_{i+1} = D_i \setminus \{v\}$ and $v, w \in D_i$. Then, again by Claim 4, there is $v' \in \{v, w\}$ such that $D_{i-1} \uplus \{v'\} = D_i$. Let $w' = \{v, w\} \setminus \{v'\}$. Since $i \geq 3$, we now apply Case 1 with respect to $w' \in D_{i-1}$ and $v' \in D_i$ to complete the proof. □

Claim 6 *For any index $j \in \{1, \ldots, k\}$ and color $q \in \{1, \ldots, k\}$, there exist an odd $i \in \{3, \ldots, \ell\}$ and $r \in \{5k, \ldots, 15k\}$ such that D_i contains a vertex of color q from $V(H^{j,r})$.*

Proof Without loss of generality, assume that $k \geq 2$. Moreover, for any odd $i \in [\ell - 2]$, there is a vertex common in D_i and D_{i+2} (since $k \geq 2$). This implies that $H[D_1 \cup D_3 \ldots D_\ell]$ is a connected subgraph of H. Notice that for any $j \in \{1, \ldots, k\}$ and $r \in [20k]$, $V(H^{(j,r)})$ is a (v_1, x_1)-separator in H. Therefore, since $H[D_1 \cup D_3 \ldots D_\ell]$ is connected and $v_1, x_1 \in D_1 \cup D_\ell$, (i) for any $j \in [k]$ and $r \in [20k]$, there is an odd $i \in [\ell]$ such that D_i contains a vertex from $V(H^{(j,r)})$. Now fix an index $j \in \{1, \ldots, k\}$ and a color $q \in \{1, \ldots, k\}$. By statement (i), there is an odd $i \in \{1, \ldots, \ell\}$ such that D_i contains a vertex from $V(H^{(j,10k)})$. Since $H[D_i]$ is connected, $|D_i| = 2k-1$, $D_i \cap V(H^{(j,10k)}) \neq \emptyset$, and any vertex in $V(H) \setminus \bigcup_{r=5k}^{15k} V(H^{(j,r)})$ is at distance more that $5k$ (by the construction of H), we have that all the vertices in D_i belong to $\bigcup_{r=5k}^{15k} V(H^{(j,r)})$. Moreover, by Claim 4, D_i contains a vertex colored q and that will also be present in $\bigcup_{r=5k}^{15k} V(H^{(j,r)})$. This completes the proof of the claim. □

Claim 7 *For any color $q \in \{1, \ldots, k\}$, the vertices of color q from $\bigcup_{i=2}^{k} V(H^i)$ used in the reconfiguration sequence $D_1, \ldots, D_\ell$ are copies of the same vertex $z \in V(G)$. Moreover, exactly one vertex from $V(H^j)$ of color q is used in the reconfiguration for all $2 \leq j \leq k$.*

Proof Fix a color $q \in \{1, \ldots, k\}$. By Claim 6, there are vertices of color q from $V(H^j)$ for all j is used in the reconfiguration sequence. By Claim 5, all these vertices are copies of the same vertex $z \in V(G)$. □

Now we define a k-size vertex subset $C \subseteq V(G)$ and prove that C is a clique in G. We let $C = \{a_i \in V(G) : 1 \leq i \leq k, c(a_i) = i$, and the copy of a_i in $V(H^2)$ is used in $D_1, \ldots, D_\ell\}$. Because of Claim 7, we have that $|C| = k$ and C contains a vertex of each color in c. $C = \{a_1, \ldots, a_k\} \subseteq V(G)$ and for each $q \in [k]$, $c(a_q) = q$. We now prove that C is indeed a clique in G. Towards that, we need to prove that for each $1 \leq q < j \leq k$, $\{a_q, a_j\} \in E(G)$.

Claim 8 *Let $1 \leq q < j \leq k$. Then, $\{a_q, a_j\} \in E(G)$.*

Proof By Claim 6, we know that there exist an odd $i \in [\ell]$ and $r \in \{5k, \ldots, 15k\}$ such that D_i contains a vertex of color q in $V(H^{(j,r)})$. Thus, by Claim 7, a copy of a_j and a copy of a_q are present in D_i. Let u_j and u_q be the vertices in D_i colored with j and q, respectively. By Claim 7, u_j is a copy of a_j and u_q is a copy of a_q. Any vertex b in $V(H^j)$ colored $k+1$ is adjacent to vertices of exactly two colors, out of which one color is j. Moreover, by the construction of H, (a) if b is adjacent to x and y in $V(H^j)$, and x and y are copies of x' and y' in G, respectively, then $\{x', y'\} \in E(G)$. We know that $H[D_i]$ is connected, $|R_s \cap D_i| = 1$ for all $1 \leq s \leq k$, $D_i \setminus R_{k+1}$ is an independent set in H, and each vertex in D_i colored with $k+1$ is adjacent to exactly two vertices in $D_i \setminus R_{k+1}$ with one of them being u_j (see Claims 1 and 4). This implies that there is common neighbor b for u_q and u_j and hence $\{a_q, a_j\} \in E(G)$, by statement (a) above. This completes the proof of the claim. □

This completes the proof of the lemma. □

Theorem 1 CCS-R *parameterized by $k + \ell$ is* W[1]*-hard on* 4*-degenerate graphs.*

3.2 Reduction from CCS-R to CDS-R.

We give a polynomial-time parameter-preserving reduction from CCS-R to CDS-R that is fairly straightforward. Let (G, c, Q_s, Q_t, k) be an instance of CCS-R. Let $c : V(G) \mapsto \{1, \ldots, k'\}$, where $k' \leq k$. We construct a graph H as follows. For each $1 \leq i \leq k'$, we add a vertex d_i and connect d_i to all the vertices in $c^{-1}(i)$. Next, for each $1 \leq i \leq k'$, we add a pendant vertex x_i (i.e., $\{d_i, x_i\}$ is an edge). Let $D = \{d_1, \ldots, d_{k'}\}$. We output $(H, Q_s \cup D, Q_t \cup D, k + k')$ as the new CDS-R instance.

Lemma 5 *If G is a d-degenerate graph then H is a $(d+1)$-degenerate graph.*

Proof For each vertex $v \in V(G)$, $d_H(v) = d_G(v) + 1$. Thus, after removing $V(G)$ and $\{x_i : 1 \leq i \leq k'\}$, the remaining graph is edgeless. □

It is easy to verify that for any reconfiguration sequence $Q_s = R_1, \ldots, R_\ell = Q_t$ of the instance (G, c, Q_s, Q_t, k) of CCS-R, $Q_s \cup D = R_1 \cup D, \ldots, R_\ell \cup D = Q_t \cup D$ is a reconfiguration sequence of the instance $(H, Q_s \cup D, Q_t \cup D, k + k')$ of CDS-R. Now we prove the reverse direction.

Lemma 6 *If* $(H, Q_s \cup D, Q_t \cup D, k+k')$ *is a* yes-*instance then* (G, c, Q_s, Q_t, k) *is a* yes-*instance.*

Proof Notice that the set D is contained in any connected dominating set of H. Moreover for any minimal connected dominating set Z in H, $Z \cap \{x_i : 1 \leq i \leq k'\} = \emptyset$, $H[Z \setminus D]$ is connected, and $Z \setminus D$ contains a vertex from $c^{-1}(i)$ for all $1 \leq i \leq k'$ (recall that G is a subgraph of H). Therefore, by deleting D from each set in a reconfiguration sequence of $(H, Q_s \cup D, Q_t \cup D, k+k')$, we get a valid reconfiguration sequence of (G, c, Q_s, Q_t, k). This completes the proof. □

Thus, by Theorem 1, we have the following theorem.

Theorem 2 CDS-R *parameterized by* $k + \ell$ *is* W[1]*-hard on* 5*-degenerate graphs.*

4 Fixed-parameter tractability on planar graphs

This section is devoted to proving that CDS-R under TAR parameterized by k is fixed-parameter tractable on planar graphs. In fact, we show that the problem admits a polynomial kernel. Recall that a kernel for a parameterized problem $\mathcal{Q}$ is a polynomial-time algorithm that computes for each instance (I, k) of $\mathcal{Q}$ an equivalent instance (I', k') with $|I'| + k' \leq f(k)$ for some computable function f. The kernel is polynomial if the function f is polynomial. We prove that for every instance (G, S, T, k) of CDS-R, with G planar, we can compute in polynomial time an instance (G', S, T, k) where $|V(G')| \leq h(k)$ for some polynomial h, G' planar, and where there exists a reconfiguration sequence under TAR from S to T in G (using at most k tokens) if and only if such a sequence exists in G'.

Our approach is as follows. We first compute a small *domination core* for G, that is, a set of vertices that captures exactly the domination properties of G for dominating sets of sizes not larger than k. While the classification of interactions with the domination core would suffice to solve Dominating Set Reconfiguration, additional difficulties arise for the connected variant. In a second step we use planarity to identify large subgraphs that have very simple interactions with the domination core and prove that they can be replaced by constant size gadgets such that the reconfiguration properties of G are preserved.

4.1 Domination cores

Definition 2 Let G be a graph and let $k \geq 1$ be an integer. A *k-domination core* is a subset $C \subseteq V(G)$ of vertices such that every set $X \subseteq V(G)$ of size at most k that dominates C also dominates G.

It is not difficult to see that DOMINATING SET is fixed-parameter tractable on all graphs that admit a k-domination core of size at most $f(k)$ that is computable in time $g(k) \cdot n^c$, for any computable functions f, g and constant c. This approach was first used (implicitly) in [5] to solve DISTANCE-r DOMINATING SET on nowhere dense graph classes. In case k is the size of a minimum (distance-r) dominating set, one can establish the existence of a linear size k-domination core on classes of bounded expansion [7] (including the class of planar graphs) and a polynomial size (in fact an almost linear size) k-domination core on nowhere dense graph classes [9, 24]. If k is not minimum, there exist classes of bounded expansion such that a k-domination core must have at least quadratic size [8]. The most general graph classes that admit k-domination cores are given in [10]. Moreover, DOMINATING SET RECONFIGURATION and DISTANCE-r DOMINATING SET RECONFIGURATION are fixed-parameter tractable on all graphs that admit small (distance-r) k-domination cores [25, 33].

Lemma 7 *There exists a polynomial h such that for all $k \geq 1$, every planar graph G admits a polynomial-time computable k-domination core of size at most $h(k)$.*

The lemma is implied by Theorem 1.6 of [24] by the fact that planar graphs are nowhere dense. We want to stress again that the polynomial size of the k-domination core results from the fact that k may not be the size of a minimum dominating set, if k is minimum we can find a linear size core. Explicit bounds on the degree of the polynomial can be derived from [30, 32], but we refrain from doing so to not disturb the flow of ideas.

The following lemma is immediate from the definition of a k-domination core.

Lemma 8 *If C is a k-domination core and D is a dominating set of size at most k that contains a vertex set $W \subset D$ such that $N[D] \cap C = N[D \setminus W] \cap C = C$, then $D \setminus W$ is also a dominating set.*

Definition 3 Let G be a graph and let $A \subseteq V(G)$. The *projection* of a vertex $v \in V(G) \setminus A$ into A is the set $N(v) \cap A$. If two vertices u, v have the same projection into A we write $u \sim_A v$.

Obviously, the relation $\sim_A$ is an equivalence relation. The following lemma is folklore, one possible reference is [11].

Lemma 9 *Let G be a planar graph and let $A \subseteq V(G)$. Then there exists a constant c such that there are at most $c \cdot |A|$ different projections to A, that is, the equivalence relation $\sim_A$ has at most $c \cdot |A|$ equivalence classes.*

4.2 Reduction rules

Let G be an embedded planar graph. We say that a vertex v *touches* a face f if v is drawn inside f or belongs to the boundary of f or is adjacent to a vertex on the boundary of f. We fix two connected dominating sets S and T of size at most k. We will present a sequence of lemmas, each of which implies a polynomial-time computable reduction rule that allows us to transform G to a planar graph G' that inherits its embedding from G, with $S, T \subseteq V(G')$ and that has the same reconfiguration properties with respect to S and T as G. To not overload notation, after stating a lemma with a reduction rule, we assume that the reduction rule is applied until this is no longer possible and call the resulting graph again G. We also assume that whenever one or more of our reduction rules are applicable, then they are applied in the order presented. We will guarantee that S and T will always be connected dominating sets of size at most k, hence, after each application of a reduction rule, we can recompute a k-domination core in polynomial time. This yields only polynomial overhead and allows us to assume that we always have marked a k-domination core C of size at most $h(k)$ as described in Lemma 7. This allows us to state the lemmas as if G and C are fixed. Without loss of generality we assume that C contains S and T.

Definition 4 A set W of vertices or edges is *irrelevant* if there is a reconfiguration sequence from S to T in G if and only if there is a reconfiguration sequence from S to T in $G - W$.

Definition 5 Let $u, v \in V(G)$ be distinct vertices. We call the set $D(u, v) := (N(u) \cap N(v)) \cup \{u, v\}$ the *diamond* induced by u and v. We call $|N(u) \cap N(v)|$ the *thickness* of $D(u, v)$.

Lemma 10 *If G contains a diamond $D(u, v)$ of thickness greater than $3k$, then at least one of u or v must be occupied by a token in every step of every reconfiguration sequence from S to T.*

Proof Assume $S = S_1, \ldots, S_t = T$ is a reconfiguration sequence from S to T and $u, v \notin S_i$ for some $1 \leq i \leq t$. Then every $s \in S_i$ can dominate at most 3

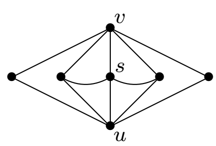


Fig. 8: A vertex $s \in S_i$ can dominate at most 3 vertices of $N(u) \cap N(v)$.

vertices of $N(u) \cap N(v)$: otherwise u, v, s together with 3 vertices of $N(u) \cap N(v)$ different from u, v and s would form a complete bipartite graph $K_{3,3}$. □

Lemma 11 *If G contains a diamond $D(u, v)$ of thickness greater than $3k$, then we can remove all internal edges in $D(u, v)$, i.e., edges with both endpoints in $N(u) \cap N(v)$.*

Proof Assume $S = S_1, \ldots, S_t = T$ is a reconfiguration sequence from S to T. According to Lemma 10, for each $1 \leq i \leq t$, $S_i \cap \{u, v\} \neq \emptyset$. Hence all vertices of $N(u) \cap N(v)$ are always dominated by at least one of u or v, say by u. Moreover, having tokens on more than one vertex of $N(u) \cap N(v)$ will never create connectivity via internal edges that is not already there via edges incident on u. In other words, for any connected dominating set S of G, if an edge yz is used for connectivity, where $y, z \in N(u) \cap N(v)$, then the edge can be replaced by the path yuz or the path yvz (depending on which of u or v is in S). □

As described earlier, we now apply the reduction rule of Lemma 11 until this is no longer possible, and name the resulting graph again G. As we did not make use of the properties of a k-domination core in the lemma, it is sufficient to recompute a k-domination core C after applying the reduction rule exhaustively. In the following it may be necessary to recompute it after each application of a reduction rule. We will not mention these steps explicitly in the following.

Lemma 12 *If G contains a diamond $D(u, v)$ of thickness greater than $4|C| + 3k + 1$ then G contains an irrelevant vertex.*

Proof Let H be the subgraph of G induced by $D(u, v)$. We enumerate the vertices of $N(u) \cap N(v)$ consecutively as $x_1, \ldots, x_t$ for some $t > 4|C| + 3k + 1$. We let $X = \{x_1, \ldots, x_t\}$. Note that since we have t vertex-disjoint paths between u and v in H, these paths define the boundaries of t faces in the plane embedding of H (after applying the reduction rule of Lemma 11, H has all the edges $\{u, x\}$ and $\{v, x\}$ for $x \in N(u) \cap N(v)$ and no other edges). Each vertex in $C \setminus \{u, v\}$ can be adjacent in H to at most two vertices in X, hence each vertex in $C \setminus \{u, v\}$ can touch at most 3 consecutive faces of H.

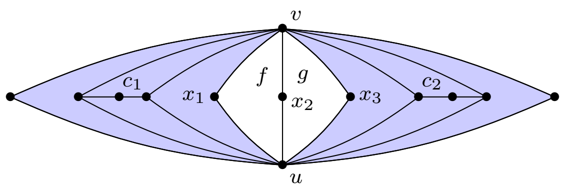


Fig. 9: Every vertex of $C \setminus \{u, v\}$ can touch at most 3 consecutive faces of H. In the figure we assume the vertices c_1 and c_2 are in $C \setminus \{u, v\}$. The faces that are touched by c_1 or c_2 are colored in blue. The uncolored faces f and g are not touched by vertices of $C \setminus \{u, v\}$.

This leaves $|C| + 3k + 1$ faces of H that are not touched by a vertex of $C \setminus \{u, v\}$. By the pigeonhole principle we can find 2 adjacent faces f and g of H that are not touched by a vertex of $C \setminus \{u, v\}$.

We let x_1 and x_2 denote the two vertices on the boundary of face f different from u and v and we let x_2 and x_3 denote the two vertices on the boundary of face g different from u and v. Recall that, due to Lemma 11, we know that there are no edges between those three vertices. Let W denote the set of all vertices contained in the face of the cycle u, x_1, v, x_3, u. In particular, W contains x_2. We claim that the vertices of W can be removed from G without changing the reconfiguration properties of G, i.e., W is a set of irrelevant vertices. Let $G' = G - W$. First observe that, since $S, T \subseteq C$, $W \cap (S \cup T) = \emptyset$, hence $S, T \subseteq V(G')$. We show that reconfiguration from S to T is possible in G if and only if reconfiguration from S to T is possible in G'.

Assume $S = S_1, \ldots, S_t = T$ is a reconfiguration sequence from S to T in G. Let $S'_1, \ldots, S'_t$, where for $1 \leq i \leq t$, $S'_i \coloneqq S_i$ if S_i does not contain a vertex of W and $S'_i \coloneqq (S_i \setminus W) \cup \{x_1\}$ otherwise. Note that this modification leaves S and T unchanged, hence, $S'_1 = S_1$ and $S'_t = S_t$. We claim that $S'_1, \ldots, S'_t$ is a reconfiguration sequence from S to T in G'.

Claim 9 *For $1 \leq i \leq t$, S'_i is a dominating set of G, and hence also of G'.*

Proof No vertex of W is adjacent to a vertex of $C \setminus \{u, v\}$ and $W \cap C = \emptyset$ by construction. Hence, the only vertices of C that are possibly adjacent to a vertex of W are the vertices u and v. Whenever S_i contains a vertex of W, we have $x_1 \in S'_i$, which dominates both u and v. Hence, S'_i dominates at least the vertices of C that S_i dominates. We use Lemma 8 to conclude that S'_i is a dominating set of G. □

Claim 10 *For $1 \leq i \leq t$, S'_i is connected.*

Proof Let $s_1, s_2 \in S_i \setminus W$ and let P be a shortest path between s_1 and s_2 in $G[S_i]$. We have to show that there exists a path between s_1 and s_2 in $G[S'_i]$. If P does not use a vertex of W, then there is nothing to show. Hence, assume P uses a vertex of W. By definition of W, both s_1 and s_2 lie outside or on the boundary of the face h of the cycle u, x_1, v, x_3 that contains x_2. Hence, P must enter and leave the face h, and as P is a shortest path, it must enter and leave via opposite vertices, i.e., via u and v, or via x_1 and x_3 (as all other pairs are linked by an edge and we could find a shorter path). If P contains u and v, then we can replace the vertices of W on P by x_1 and we are done.

Hence, assume P uses x_1 and x_3. As $D(u, v)$ is a diamond of thickness greater than $4|C| + 3k + 1 > 3k$, according to Lemma 10 at least one of the vertices u and v, say u, is contained in S_i, and by definition also in S'_i. Then we can replace the vertices of W on P by u and we are again done. □

Finally, the following claim is immediate from the definition of each S'_i. Combining Claims 9, 10, and 11, we conclude that $S'_1, \ldots, S'_t$ is a reconfiguration sequence from S to T in G'.

Claim 11 *S'_{i+1} is obtained from S'_i by the addition or removal of a single token for all $1 \leq i < t$.*

To prove the opposite direction, assume $S = S'_1, \dots, S'_t = T$ is a reconfiguration sequence from S to T in G'. We claim that this is also a reconfiguration sequence from S to T in G. All we have to show is that S'_i is a dominating set of G for all $1 \leq i \leq t$. This follows immediately from the fact that S'_i is a dominating set of G', and hence, as W is not adjacent to $C \setminus \{u, v\}$ and $W \cap C = \emptyset$, also a dominating set of C in G. Then according to Lemma 8, S'_i also dominates G. We conclude that there is a reconfiguration sequence from S to T in G if and only if there is a reconfiguration sequence from S to T in $G' = G - W$. □

We may in the following assume that G does not contain diamonds of thickness greater than $4|C| + 3k + 1$.

Corollary 1 *If a vertex $v \in V(G)$ has degree greater than $(4|C| + 3k + 1) \cdot k$, then the token on v is never lifted throughout a reconfiguration sequence.*

Proof Assume $S = S_1, \dots, S_t = T$ is a reconfiguration sequence from S to T in G and assume there is S_i with $v \notin S_i$. The dominating set S_i has at most k vertices and must dominate $N(v)$. Hence, there must be one vertex $u \in S_i$ that dominates at least a $1/k$ fraction of $N(v)$, which is larger than $4|C| + 3k + 1$. Then there is a diamond $D(u, v)$ of thickness greater than $4|C| + 3k + 1$, which does not exist after application of the reduction rule of Lemma 12. □

According to Corollary 1, the only vertices that can have high degree after applying the reduction rules are vertices that are never lifted throughout a reconfiguration sequence. This gives rise to another reduction rule that is similar to the rule of Lemma 11.

Lemma 13 *Assume v is a vertex of degree greater than $(4|C| + 3k + 1) \cdot k$. Then we may remove all edges with both endpoints in $N(v)$.*

Proof Let G' be the graph obtained from G by removing all edges with both endpoints in $N(v)$. We claim that reconfiguration between S and T is possible in G if and only if it is possible in G'. The fact that S and T are in fact connected dominating sets in G' is implied by the argument below.

Assume $S = S_1, \dots, S_t = T$ is a reconfiguration sequence from S to T in G. We claim that the same sequence is a reconfiguration sequence in G'. According to Corollary 1, $v \in S_i$ for all $1 \leq i \leq t$. This implies that S_i is connected in G' for all $1 \leq i \leq t$, as all $x, y \in S_i$ that are no longer connected by an edge in G' but were connected in G are connected via a path of length 2 using the vertex v. It is also easy to see that S_i is a dominating set in G', as all vertices that are no longer dominated by $s \in S_i$ in G are still dominated by v. Observe that this in particular implies that S and T are connected dominating sets in G'. Vice versa, if $S = S_1, \dots, S_t = T$ is a reconfiguration sequence from S to T in G', this is trivially also a reconfiguration sequence in G. □

The following reduction rule is obvious.

Lemma 14 *If a vertex v has more than $k+1$ pendant neighbors, i.e., neighbors of degree exactly one, then it suffices to retain exactly $k+1$ of them in the graph.*

Lemma 15 *There are at most $c|C| \cdot (4|C| + 3k + 1)$ vertices of $V(G) \setminus C$ that have 2 neighbors in C, where c is the constant of Lemma 9.*

Proof According to Lemma 9 there are at most $c|C|$ different projections to C. Each projection class that has at least 3 representatives has size at most 2, as otherwise we would find a $K_{3,3}$ as a subgraph, contradicting the planarity of G. Consider a class with a projection of size 2 into C. Denote these two vertices of C by u and v. If this class has more than $4|C| + 3k + 1$ representatives, then $D(u, v)$ is a diamond of thickness greater than $4|C| + 3k + 1$, which cannot exist after exhaustive application of the reduction rule of Lemma 12. □

We now come to the description of our final reduction rule. Let D denote the set of vertices containing both C and all vertices of $V(G) \setminus C$ having at least two neighbors in C. In other words, $V(G) \setminus D$ contains all those vertices in $V(G) \setminus C$ that have exactly one neighbor in C. According to Lemma 15 at most $c|C| \cdot (4|C| + 3k + 1)$ vertices have two neighbors in C, hence $|D| \leq c|C| \cdot (4|C| + 3k + 1) + |C| =: p$.

Lemma 16 *Assume there are two vertices u and v with degree greater than $4p + (4|C| + 3k + 1) \cdot k + 1$. Let $\mathcal{P}$ be a maximum set of vertex-disjoint paths of length at least 2 that run between u and v using only vertices in $V(G) \setminus D$. If $|\mathcal{P}| > 4p + (4|C| + 3k + 1) \cdot k + 1$, then there is G' such that the instances (G, S, T, k) and (G', S, T, k) are equivalent, G' is planar, and $|V(G')| < |V(G)|$.*

Proof We first show that we can essentially establish the situation depicted in Figure 10. We may assume that the paths of $\mathcal{P}$ are induced paths, otherwise we may replace them by induced paths. Let H be the graph induced on u, v and the vertices of $\mathcal{P}$. In the figure, the paths of $\mathcal{P}$ are depicted by thick edges, while the diagonal edges do not belong to the paths. This situation is similar to the situation in the proof of Lemma 12. Just as in the proof of Lemma 12, we find two adjacent faces f, g of H that do not touch a vertex of $D \setminus \{u, v\}$.

Claim 12 *The paths bounding f and g have length 3, i.e., they have exactly two inner vertices.*

Proof First observe that $P \in \mathcal{P}$ cannot have length exactly 2, as then P contains a vertex adjacent to both u and v. However, the vertices with this property lie in D, and hence by construction not on P.

Assume there is $P \in \mathcal{P}$ of length greater than 3. Let $M(u)$ denote the neighbors of u that are in $V(G) \setminus D$ and are only adjacent to u and to no other vertex of C. Similarly, let $M(v)$ denote the neighbors of v that are in $V(G) \setminus D$ and are only adjacent to v and to no other vertex of C. By construction, the faces f and g do not contain vertices of $D \setminus \{u, v\}$. Furthermore, P contains exactly one vertex of $M(u)$ and exactly one vertex of $M(v)$. It cannot contain two vertices

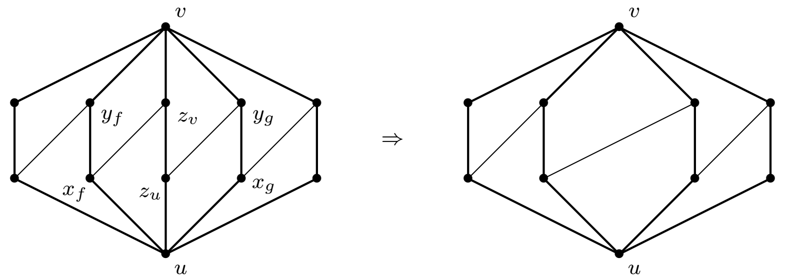


Fig. 10: An exemplary situation handled by Lemma 16.

of one of these sets, as otherwise P is not an induced path. Hence, assume that P contains another vertex x that is not in $M(u) \cup M(v)$. Then x must be dominated by a vertex different from u and from v. However, by construction, the faces f and g do not touch a vertex of $D \setminus \{u, v\} \supseteq (S \cup T) \setminus \{u, v\}$, a contradiction. □

Denote by x_f, y_f the two vertices that lie on the boundary of f and not on the boundary of g and by x_g, y_g the two vertices that lie on the boundary of g and not on the boundary of f. Assume that $x_f, x_g \in M(u)$ and $y_f, y_g \in M(v)$. Denote by z_u, z_v the vertices shared by f and g different from u and v that are adjacent to u and v, respectively. Denote by W the set of all vertices that lie inside the face h of the cycle $u, x_f, y_f, v, y_g, x_g, u$ that contains the vertices z_u and z_v. Hence W contains at least the vertices z_u and z_v. By Corollary 1, we know that $u, v \in S_i$, for all $1 \leq i \leq t$ (both u and v can never be lifted). Consequently, by Lemma 13, we know that there are no edges with both endpoints in $N(v)$ nor edges with both endpoints in $N(u)$. Combining the previous fact with the fact that all vertices of W are adjacent to either u or v (but not both) and to no other vertex of $C \supseteq S \cup T$, we conclude that W consists of exactly the two vertices z_u and z_v and that there are no edges between z_u and x_g, x_f and no edges between z_v and y_g, y_f. Note that we can safely assume that none of the degree-one neighbors of u or v are inside W. We claim that the vertices z_u and z_v are irrelevant and can be removed after possibly introducing an additional edge to the graph. Recall that S and T do not contain the vertices z_u and z_v. We define G' as follows.

- If $\{u, v\} \notin E(G)$ and ($\{x_f, z_v\} \in E(G)$ or $\{y_f, z_u\} \in E(G)$) and ($\{x_g, z_v\} \in E(G)$ or $\{y_g, z_u\} \in E(G)$) then G' is obtained from G by deleting z_u and z_v and introducing the edge $\{x_f, y_g\}$.
- Otherwise, G' is obtained from G by simply deleting z_u and z_v.

We claim that (G, S, T, k) and (G', S, T, k) are equivalent instances of CDS-R. Assume first that there exists a reconfiguration sequence $S = S_1, \ldots, S_t = T$ in G. We distinguish two cases. First assume that $\{u, v\} \in E(G)$. Hence, G' is obtained from G by simply deleting z_u and z_v. Let $S'_1, \ldots, S'_t$, where for

$1 \leq i \leq t$, $S'_i = S_i \setminus \{z_u, z_v\}$. We claim that $S'_1, \ldots, S'_t$ is a reconfiguration sequence from S to T in G'.

Claim 13 *For $1 \leq i \leq t$, S'_i is a dominating set of G, and hence also of G'.*

Proof The vertices z_u and z_v are not adjacent to a vertex of $C \setminus \{u, v\}$ and $\{z_u, z_v\} \cap C = \emptyset$. Hence, the only vertices of C that are possibly adjacent to z_u or z_v are the vertices u and v. According to Lemma 1, $u, v \in S_i$, and moreover $u, v \in S'_i$, for all $1 \leq i \leq t$. Hence, S'_i dominates at least the vertices of C that S_i dominates. We use Lemma 8 to conclude that S'_i is a dominating set of G. □

Claim 14 *For $1 \leq i \leq t$, S'_i is connected.*

Proof Let $s_1, s_2 \in S_i \setminus \{z_u, z_v\}$ and let P be a shortest path between s_1 and s_2 in $G[S_i]$. We have to show that there exists a path between s_1 and s_2 in $G[S'_i]$. If P does not use z_u nor z_v then there is nothing to prove. Hence, assume P uses z_u or z_v (or both). By definition of W, both s_1 and s_2 lie outside the face h of the cycle $u, x_f, y_f, v, y_g, x_g, u$ that contains z_u, z_v. Hence, P must enter and leave the face h, say it enters at u and leaves at y_f. All other possibilities are handled analogously. Then we can avoid the vertices z_u and z_v by walking to v first, then u (or x_f), and then to y_f. □

The next claim follows from the definition of S'_i and the fact that we can remove any duplicate consecutive sets in a reconfiguration sequence.

Claim 15 *S'_{i+1} is obtained from S'_i by the addition or removal of a single token for all $1 \leq i < t$.*

This finishes the proof in case $\{u, v\} \in E(G)$. Hence, we assume now that $\{u, v\} \notin E(G)$ and $(\{x_f, z_v\} \in E(G)$ or $\{y_f, z_u\} \in E(G))$ and $(\{x_g, z_v\} \in E(G)$ or $\{y_g, z_u\} \in E(G))$. That is, G' is obtained from G by deleting z_u and z_v and introducing the edge $\{x_f, y_g\}$. We now obtain S'_i from S_i, for $1 \leq i \leq t$, by replacing

- z_u by x_f and z_v by y_g if $S_i \cap \{z_u, z_v\} = \{z_u, z_v\}$,
- z_u by x_f if $S_i \cap \{z_u, z_v\} = \{z_u\}$, and
- z_v by y_g if $S_i \cap \{z_u, z_v\} = \{z_v\}$.

We claim that $S'_1, \ldots, S'_t$ is a reconfiguration sequence from S to T in G'. We need no new arguments to prove that each S'_i is a dominating set of G and hence of G' and that each S'_{i+1} is obtained from S'_i by adding or removing one token. It remains to show that each S'_i is connected in G'.

Claim 16 *For $1 \leq i \leq t$, S'_i is connected in G'.*

Proof According to Lemma 1, $u, v \in S_i$, and also $u, v \in S'_i$, for all $1 \leq i \leq t$. If $S_i \setminus \{z_u, z_v\}$ is connected, S'_i is also connected, hence assume $S_i \setminus \{z_u, z_v\}$ is not connected. As $X = \{u, x_f, z_u, x_g\}$ is connected via u and $Y = \{v, y_f, z_v, y_g\}$ is connected via v, it suffices to show that our vertex exchange creates a connection in G' between any vertex of X and any vertex of Y. If $S_i \cap \{z_u, z_v\} = \{z_u, z_v\}$ this is clear, as we shift the tokens to x_f and y_g and in G' we have introduced the edge $\{x_f, y_g\}$. If $S_i \cap \{z_u, z_v\} = \{z_u\}$, then $\{z_u, y_g\} \in E(G)$ and $y_g \in S_i$, or $\{z_u, y_f\} \in E(G)$ and $y_f \in S_i$. We move the token z_u to x_f. In the first case we have connectivity via the new edge $\{x_f, y_g\} \in E(G')$, and in the second case we have connectivity via the edge $\{x_f, y_f\} \in E(G)$. The case $S_i \cap \{z_u, z_v\} = \{z_v\}$ is symmetric. □

This finishes the proof that if (G, S, T, k) is a positive instance then (G', S, T, k) is a positive instance. Now assume that there exists a reconfiguration sequence $S = S'_1, \ldots, S'_t = T$ in G'. In case we do not introduce the new edge to obtain G' from G, we do not need new arguments to see that $S'_1, \ldots, S'_t$ is a reconfiguration sequence also in G. Moreover, if $G''[S'_i]$ is connected for all i, where G'' is obtained from G' by removing the edge $\{x_f, y_g\}$, then again there is nothing to prove as G' is a subgraph of G and therefore $S = S'_1, \ldots, S'_t = T$ is a reconfiguration sequence in G. Hence, assume that there exists at least one contiguous subsequence σ starting at index s and ending at index f (with possibly $s = f$) such that $G''[S'_s], G''[S'_{s+1}], \ldots, G''[S'_f]$ are not connected. In other words, there exists a subsequence of length one or more that uses the edge $\{x_f, y_g\}$ for connectivity. Moreover, we assume, without loss of generality (the other case is symmetric), that S'_s is obtained from S'_{s-1} by adding a token on vertex y_g, i.e., $S'_s = S'_{s-1} \cup \{y_g\}$, and S'_{f+1} is obtained from S'_f by removing the token on vertex x_f, i.e., $S'_{f+1} = S'_f \setminus \{x_f\}$. We also assume that $E(G)$ contains the edges $\{x_f, z_v\}$ and $\{z_u, y_g\}$ (the remaining cases are handled identically). It remains to show how to modify σ so that it does not use the edge $\{x_f, y_g\}$ for connectivity and remains a valid reconfiguration sequence in G. By applying the same arguments for any such subsequence we obtain the required reconfiguration sequence in G. We modify σ as follows. We let $S''_i = (S'_i \setminus \{y_g\}) \cup \{z_v\}$, for $s \leq i \leq f$. Then we replace S'_{f+1} by four new sets A_1, A_2, A_3, and A_4, where $A_1 = S'_f \setminus \{x_f\}$, $A_2 = A_1 \cup \{z_u\}$, $A_3 = A_2 \setminus \{z_v\}$, $A_3 = A_3 \cup \{y_g\}$, and $A_4 = A_3 \setminus \{z_u\}$. Using the fact that the vertices x_f, y_f, x_g, y_g are not adjacent to vertices of $D \setminus \{u, v\}$, it is easy to see that this yields a valid reconfiguration sequence, as both domination and connectivity are preserved. This completes the proof of the lemma. □

We are ready to state the final result.

Theorem 3 CDS-R *under TAR parameterized by k admits a polynomial kernel on planar graphs.*

Proof Our kernelization algorithm starts by computing (in polynomial time) a k-domination core C of size at most $h(k)$ as described in Lemma 7. Without loss of generality we assume that C contains S and T. After each application of a reduction rule, we recompute the core, giving a polynomial blow-up of the running time. We are left to prove that each reduction rule can be implemented in polynomial time and that we end up with a polynomial number of vertices. It is clear that the reduction rules of Lemma 12, Lemma 13 and Lemma 14 can easily be implemented in polynomial time. The reduction rule of Lemma 16 is slightly more involved, however, we can use a standard maximum-flow algorithm to compute in polynomial time a maximum set of vertex-disjoint paths in a subgraph of G. It remains to bound the size of G. Recall that we call D the set of all vertices C and of all vertices of $V(G) \setminus C$ that have at least 2 neighbors in C. It follows from Lemma 15 that D has size at most $c|C| \cdot (4|C|+3k+1)+|C| =: p$, where c is the constant of Lemma 9. We are left to bound the number of vertices in $V(G) \setminus C$ having exactly one neighbor in C (recall that each vertex in $V(G) \setminus C$ has at least one neighbor in $S \cup T \subseteq C$).

Let $p' = (4p + (4|C| + 3k + 1) \cdot k + 1) \cdot (4|C| + 3k + 1) \cdot k + k + 1$, which is still a polynomial in k. Towards a contradiction, assume that there exists an equivalence class Q in $\sim_C$ with a projection of size one containing more than p' vertices. Let $u \in C$ denote the projection of the aforementioned class. Due to Lemma 14, we know that at most $k+1$ of the vertices in Q are pendant, i.e., adjacent to only u in G. Since we cannot apply the reduction rule of Lemma 13 any more, we know that there are no edges with both endpoints in Q. Hence, all but $k + 1$ vertices of Q must be adjacent to at least one other vertex in $V(G) \setminus C$. Let $R = N_G(Q) \setminus \{u\}$ denote this set of neighbors. No vertex in R can be adjacent to more than $4|C| + 3k + 1$ vertices of Q, as we cannot apply the reduction rule of Lemma 12. The vertices of R must be dominated by S, and cannot be dominated by u, as otherwise two neighbors of u would be connected. Hence, there is $v \in S$ different from u that dominates at least a $1/k$ fraction of R. This implies the existence of at least $4p + (4|C| + 3k + 1) \cdot k + 1$ vertex-disjoint paths of length at least 2 that run between u and v. But in this case, the reduction rule of Lemma 16 is applicable. Therefore, we conclude that Q cannot exist, obtaining a bound on the size of all equivalence classes of $\sim_C$, as needed. ◻

5 Conclusion

We have shown that the CDS-R problem parameterized by k is fixed-parameter tractable for planar graphs and (trivially) for graphs of bounded degree. Moreover, a simple observation shows that the problem is fixed-parameter tractable parameterized by $k + \ell$ on every nowhere dense graph class and the same holds for every class of bounded cliquewidth. On the negative side, our reduction shows that CDS-R parameterized by $k+\ell$ is W[1]-hard on 5-degenerate graphs. It remains open to determine where exactly the boundary between tractable and intractable lies for CDS-R parameterized by k. We conjecture that CDS-R is fixed-parameter tractable parameterized by k on every nowhere dense graph class. However, resolving this conjecture remains open for future work (see Figure 2). Towards proving that conjecture, we believe that the classes of graphs of bounded pathwidth or treewidth are the obvious next classes to study.

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