The structure of k-Lucas cubes*

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Abstract

Fibonacci cubes and Lucas cubes have been studied as alternatives for the classical hypercube topology for interconnection networks. These families of graphs have interesting graph theoretic and enumerative properties. Among the many generalization of Fibonacci cubes are k-Fibonacci cubes, which have the same number of vertices as Fibonacci cubes, but the edge sets determined by a parameter k. In this work, we consider k-Lucas cubes, which are obtained as subgraphs of k-Fibonacci cubes in the same way that Lucas cubes are obtained from Fibonacci cubes. We obtain a useful decomposition property of k-Lucas cubes which allows for the calculation of basic graph theoretic properties of this class, such as the number of edges, the average degree of a vertex, the number of hypercubes they contain, the diameter and the radius.

Keywords: Hypercube, Fibonacci cube, Lucas cube, k-Fibonacci cube, Fibonacci number, Lucas number.
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1 Introduction

An n-dimensional hypercube Q_n is the graph whose vertices are the all binary strings of length n, adjacent when their string representations differ in exactly one position. Hypercubes are one of the basic models for interconnection networks. In [3] and [12] Fibonacci cubes Γ_n and Lucas cubes Λ_n were defined as alternative topologies for the interconnection networks. Both of these networks are special subgraphs of Q_n with interesting properties.

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A binary string $b_1 b_2 \ldots b_n$ such that $b_i \cdot b_{i+1} = 0$ for $1 \leq i \leq n-1$ is called a Fibonacci string of length n. For $n \geq 1$ the Fibonacci cube Γ_n is the subgraph of Q_n induced by vertices indexed by the Fibonacci strings of length n. By convention $\Gamma_0 = Q_0$. By removing all the vertices that start and end with 1 from the vertex set of Γ_n, Lucas cubes Λ_n are obtained. This additional requirement corresponds to the Fibonacci strings $b_1 b_2 \ldots b_n$ also satisfying $b_1 \cdot b_n = 0$ for $n \geq 2$.

Graph theoretic and enumerative properties of Fibonacci cubes and Lucas cubes have been extensively studied in the literature. A survey of the some of the properties of Γ_n is presented in [7]. Basic graph theoretic properties of Λ_n appear in [12]. The average degree of a vertex in Γ_n and Λ_n are computed in [9] and the induced d-dimensional hypercubes Q_d in Γ_n and Λ_n are studied in [8, 2, 13, 14, 10, 15].

There are also other variants of interest inspired by these families of graphs. In [4] and [5], the generalized Fibonacci cube $Q_n(f)$ and the generalized Lucas cube $Q_d(\overleftarrow{f})$ are defined by removing all the vertices that contain some forbidden string f, and by removing all vertices that have a circular rearrangement containing f as a substring, respectively. With this formulation one has $\Gamma_n = Q_n(11)$ and $\Lambda_n = Q_d(\overleftarrow{f})$. The matchable Lucas cubes and their basic properties are studied in [16]. A new family of graphs akin to the Fibonacci cubes called Pell graphs are introduced in [11]. The k-Fibonacci cubes Γ_n^k which are obtained by eliminating certain edges from Γ_n are considered in [1] (see, Section 2 also).

In this work, we consider the subgraph of Γ_n^k which is obtained by removing all the vertices that start and end with 1. The idea is analogous to the construction of Λ_n from Γ_n and $Q_d(\overleftarrow{f})$ from $Q_d(f)$. The graphs Λ_n^k we obtain from Γ_n^k (called k-Lucas cubes) depend on a parameter k just like k-Fibonacci cubes. We obtain graph theoretic properties of k-Lucas cubes such as the number of edges, the average degree of a vertex, the number of induced hypercubes, the diameter and the radius.

2 Preliminaries

Fibonacci numbers and Lucas numbers are defined by the same recursion $f_n = f_{n-1} + f_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $f_0 = 0$, $f_1 = 1$; $L_0 = 2$ and $L_1 = 1$. Using the Zeckendorf or canonical representation, it is known that any positive integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers. For a given positive integer i with $0 < i \leq f_{n+2} - 1$ writing $i = \sum_{j=1}^{n} b_j \cdot f_{n-j+2}$, where $b_j \in \{0, 1\}$ and no two consecutive b_j's are 1. $(b_1, b_2, \ldots, b_n)$ gives the Zeckendorf representation of i corresponding to the Fibonacci string $b_1 b_2 \ldots b_n$. We assume that 0 has Zeckendorf representation $(0, 0, \ldots, 0)$.

The distance between two vertices u and v in a connected graph G is defined as the length of a shortest path between u and v in G. For Q_n, Γ_n and Λ_n this distance coincides with the Hamming distance d_H, which is the number of different bits in the binary string representation of the vertices. Let $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Then the vertex set and the edge set of Γ_n and Λ_n can be written as

$$\begin{aligned} V(\Gamma_n) &= \{b_1 b_2 \dots b_n \mid b_i \in \{0,1\} \text{ with } b_i \cdot b_{i+1} = 0, 1 \le i < n\} \\ E(\Gamma_n) &= \{\{u,v\} \mid u, v \in V(\Gamma_n) \text{ and } d_H(u,v) = 1\} \end{aligned}$$

and

$$\begin{aligned} V(\Lambda_n) &= \{b_1 b_2 \dots b_n \mid b_i \in \{0,1\} \text{ with } b_i \cdot b_{i+1} = 0, 1 \le i < n \text{ and } b_1 \cdot b_n = 0\} \\ E(\Lambda_n) &= \{\{u,v\} \mid u, v \in V(\Lambda_n) \text{ and } d_H(u,v) = 1\}. \end{aligned}$$

Note that the number of vertices of Γ_n is f_{n+2} and the number of vertices of Λ_n is L_n.

Γ_n can be decomposed into the subgraphs induced by the vertices that start with 0 and 10 respectively. The vertices that start with 0 constitute a graph isomorphic to Γ_{n-1} and the vertices that start with 10 constitute a graph isomorphic to Γ_{n-2}. This can be written symbolically as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2} \tag{1}$$

and usually referred to as the fundamental decomposition [7] of Γ_n. In (1), there is a matching between $10\Gamma_{n-2}$ and its copy $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$. We call the f_n edges of the matching between $10\Gamma_{n-2}$ and $00\Gamma_{n-2}$ *link edges*. Since reversal $b_1 b_2 \dots b_n \to b_n \dots b_2 b_1$ is an automorphism of Γ_n, the decomposition can also be written in the form

$$\Gamma_n = \Gamma_{n-1}0 + \Gamma_{n-2}01 \ .$$

Using these decompositions of Γ_n we can write

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2} = 0\Gamma_{n-1} + \big(10\Gamma_{n-3}0 + 10\Gamma_{n-4}01\big),$$

and consequently

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0 \ . \tag{2}$$

Note that in the decomposition (2) of Λ_n in terms of Fibonacci cubes, there are f_{n-1} link edges between $10\Gamma_{n-3}0$ and its copy $00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$.

2.1 k-Fibonacci cubes

In this section we recall some of the basic properties of k-Fibonacci cubes introduced in [1].

Let $\Gamma_0^k = \Gamma_0$ and $\Gamma_1^k = \Gamma_1$. For $n \geq 2$, Γ_n^k is defined in terms of Γ_{n-1}^k and Γ_{n-2}^k, in a manner that is similar to the fundamental decomposition of Γ_n. The difference is as follows: Instead of the f_n link edges that exist between $10\Gamma_{n-2}$ and its copy $00\Gamma_{n-2}$ in Γ_{n-1}, in the construction of Γ_n^k from $10\Gamma_{n-2}^k$ and its copy $00\Gamma_{n-2}^k$ in $0\Gamma_{n-1}^k$, there are only k link edges between the first k vertices with labels $0, 1, \ldots, k-1$ in $00\Gamma_{n-2}^k$ and the vertices with labels $f_n, f_n+1, \ldots f_n+k-1$ in $10\Gamma_{n-2}^k$. In Figure 1, we illustrate the constructions of Γ_4^1 and Γ_4^2 from the previous k-Fibonacci cubes. Note that in Figure 1, there is only one link edge between the vertices having labels 0000 and 1000 in Γ_4^1 as $k=1$ and there are two link edges between the vertices having labels 0000 and 1000; 0001 and 1001 in Γ_4^2 as $k=2$.

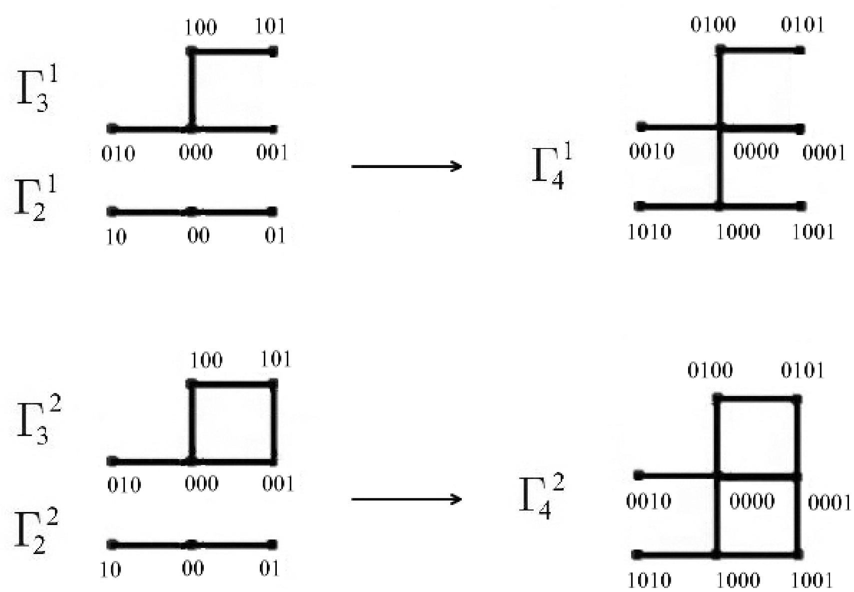

Figure 1: Construction of the k-Fibonacci cubes Γ_4^1 and Γ_4^2.

By definition, we have $\Gamma_n^k = \Gamma_n$ for $f_n \leq k$. Let $n_0(k)$ be the smallest integer for which $f_{n_0(k)} > k$. For a given k, $n_0(k)$ is the smallest integer n for which $\Gamma_n^k \neq \Gamma_n$. It can be shown that

$$n_0(k) = 1 + \left\lfloor \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2} \right) \right\rfloor$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This sequence starts as

$$3, 4, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 10, \ldots$$

If k is clear from the context we will use n_0 for $n_0(k)$.

3 k-Lucas cubes

In this section we introduce k-Lucas cubes, a special subgraph of k-Fibonacci cubes. We will indicate the dependence on k by a superscript and denote these graphs by Λ_n^k. Similar to the definition of Λ_n as the subgraph of Γ_n obtained by eliminating the vertices with $b_1 = b_n = 1$, we define the k-Lucas cube Λ_n^k from the k-Fibonacci cube Γ_n^k by eliminating the vertices with $b_1 = b_n = 1$. In other words, Λ_n^k is obtained from Γ_n^k as the induced subgraph of Γ_n^k in which the binary labels of the vertices satisfy the additional requirement $b_1 \cdot b_n = 0$.

For $k = 1$, the graphs Λ_n^1 are all trees. Of course the number of vertices in Λ_n^1 is $|E(\Lambda_n^1)| = L_n$. The height h_n of Λ_n^1 satisfies $h_1 = 0$, $h_2 = 1$ and $h_n = \min\{h_{n-1}, 1 + h_{n-2}\}$. Therefore the height of the tree with the 0 vertex as the root is given by $h_n = \lfloor n/2 \rfloor$. Figure 2 shows the first five k-Lucas cubes (trees) for $k = 1$.

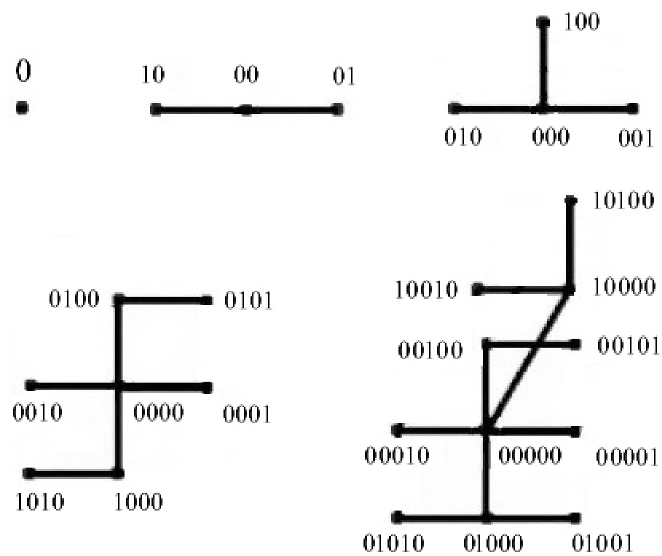

Figure 2: The first five k-Lucas cubes $\Lambda_1^1, \Lambda_2^1, \ldots, \Lambda_5^1$ for $k = 1$.

Recall that for a given k, $n_0(k)$ is the smallest integer n for which $\Gamma_n^k \neq \Gamma_n$. By definition of Λ_n^k and Γ_n^k, $n_0(k)$ is again the smallest integer n for which $\Lambda_n^k \neq \Lambda_n$, except when $k = 1$. From Figure 2 one can see that $\Lambda_n^1 \neq \Lambda_n$ for $n \geq 4 = n_0(1) + 1$.

By removing the vertex having label 1001 from Γ_4 and Γ_4^2 shown in Figure 1, we obtain the Lucas cube Λ_4 and the 2-Lucas cube Λ_4^2 given in Figure 3.

The first eight k-Lucas cubes $\Lambda_1^k, \Lambda_2^k, \ldots, \Lambda_8^k$ for the values $k = 1, 3, 6$ and 12 are presented in the Appendix.

We start with a useful result that we need for the analysis of k-Lucas cubes.

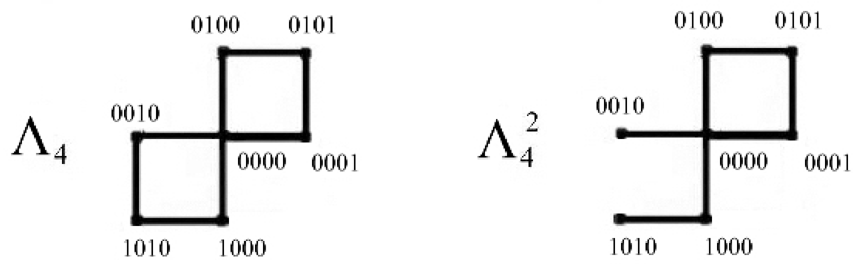

Figure 3: The Lucas cube Λ_4 and the 2-Lucas cube Λ_4^2. Λ_4 is obtained from the Fibonacci cube Γ_4 by eliminating the vertex labeled 1001 and Λ_4^2 is obtained from Γ_4^2 of Figure 1 by eliminating the vertex 1001.

Lemma 1. *Given positive integers c and k, the number of integers N with $c < N < c + k$ whose Zeckendorf representation $b_1 b_2 \ldots b_r$ satisfies $b_r = 1$ is given by*

$$\left\lfloor \frac{k+1}{\phi^2} \right\rfloor \tag{3}$$

where ϕ is the golden ratio.

Proof. By a simple translation, it suffices to prove this for $c = 0$. The integers $N > 0$ with $b_r = 1$ are those with "odd" Zeckendorf expansions. This sequence $1, 4, 6, 9, 12, 14, 17, \ldots$ forms the first column of the Wythoff array [6], and its mth term is given explicitly by

$$\lfloor \phi^2 m \rfloor - 1 \ .$$

Therefore for the lemma we need to count the the number of m satisfying the inequalities

$$0 < \lfloor \phi^2 m \rfloor - 1 < k \ .$$

The lemma follows immediately by the properties of the floor function. $\square$

For the rest of the paper for a given positive integer k we will always assume that

$$\ell = \ell(k) = k - \left\lfloor \frac{k+1}{\phi^2} \right\rfloor . \tag{4}$$

Next we consider a decomposition for Λ_n^k that will be useful in our calculations.

Theorem 1. *Let ℓ be as in (4). The k-Lucas cube Λ_n^k has the decomposition*

$$\Lambda_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-3}^\ell 0$$

in which there are ℓ link edges between $10\Gamma_{n-3}^\ell 0$ and its copy $00\Gamma_{n-3}^\ell 0 \subset 0\Gamma_{n-1}^k$.

Proof. From the fundamental decomposition of k-Fibonacci cubes, we can write $\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k$ with k link edges between the vertices with labels $0, \ldots, k-1$ in $0\Gamma_{n-1}^k$ and the corresponding vertices with labels $f_{n+1}, \ldots, f_{n+1}+k-1$ in $10\Gamma_{n-2}^k$. Now we consider the effect of eliminating all vertices in Γ_n^k which start and end with 1. This elimination has no effect on $0\Gamma_{n-1}^k$, so all of these vertices are also in Λ_n^k. For $10\Gamma_{n-2}^k$, we need to consider which vertices survive in this subgraph itself, how does the elimination change this graph, and in addition the effect of this elimination on the original k link edges. Any link edge of the original Γ_n^k which has an end vertex in $10\Gamma_{n-2}^k$ which has been eliminated, is no longer a link edge in Λ_n^k. From Lemma 1 with $c = f_{n+1}$, we know that the number of the first k vertices in $10\Gamma_{n-2}^k$ that end with 1 is given by (3). Therefore only ℓ of the original link edges survive.

f_{n-1} of the vertices in $10\Gamma_{n-2}^k$ end with 0 and f_{n-2} of them end with 1. For Λ_n^k the f_{n-2} ending with 1 are removed. Now $10\Gamma_{n-2}^k \subseteq 10\Gamma_{n-2} = 10\Gamma_{n-3}0 + 10\Gamma_{n-4}01$. Therefore, after removing the f_{n-2} vertices ending with 1 from $10\Gamma_{n-2}^k$, this has the effect of reducing the number of the link edges that appear in the construction of this graph itself to ℓ. In other words, the resulting graph is $10\Gamma_{n-3}^\ell 0 \subseteq 10\Gamma_{n-3}0$. This completes the proof. □

Example 1. *Consider Λ_6^2 obtained from Γ_6^2. We have the decomposition of Γ_6^2 as*

$$\Gamma_6^2 = 0\Gamma_5^2 + 10\Gamma_4^2 \ .$$

The link edges in Γ_6^2 are between the vertices labeled $000000, 000001$ in $0\Gamma_5^2$, and 100000, 100001 respectively in $10\Gamma_4^2$. Of these two link edges, the second one is eliminated because the vertex 100001 is not in Λ_6^2. We note that the vertices labeled $100001, 100101, 101001$ are eliminated from $10\Gamma_4^2$ in the construction of Λ_6^2. In this case $\ell = 1$ and the subgraph of $10\Gamma_4^2$ obtained after the elimination of these vertices is isomorphic to Γ_3^1, which gives $\Lambda_6^2 = 0\Gamma_5^2 + 10\Gamma_3^1 0$.

Similar to the proof of Theorem 1, we obtain the following decomposition of Γ_n^k which we state here for the record.

Corollary 1. *k-Fibonacci cube Γ_n^k has the decomposition*

$$\Gamma_n^k = \Gamma_{n-1}^\ell 0 + \Gamma_{n-2}^{k-\ell} 01$$

where ℓ is as in (4), Γ_{n-2}^0 is the graph with f_n vertices and no edges and there is a matching between $\Gamma_{n-2}^{k-\ell}01$ and $\Gamma_{n-2}^{k-\ell}00 \subset \Gamma_{n-1}^\ell 0$.

4 Basic properties of k-Lucas cubes Λ_n^k

By definition of Λ_n^k we know that $|V(\Lambda_n^k)| = |V(\Lambda_n)| = L_n$. Next we consider basic graph theoretical parameters associated with k-Lucas cubes.

4.1 The number of edges

Let $m(G) = |E(G)|$ denote the number of edges of G. It is shown in [12] that $m(\Lambda_n) = nf_{n-1}$ for $n \geq 1$. Since $m(\Lambda_n^k) = m(\Lambda_n)$ for $n < n_0$, we have $m(\Lambda_n^k) = nf_{n-1}$ for $1 \leq n < n_0$.

From Theorem 1 we observe that $m(\Lambda_n^k)$ satisfies

$$m(\Lambda_n^k) = m(\Gamma_{n-1}^k) + m(\Gamma_{n-3}^\ell) + \min\{\ell, f_{n-1}\}\ , \tag{5}$$

and for $n \geq n_0$, (5) reduces to

$$m(\Lambda_n^k) = m(\Gamma_{n-1}^k) + m(\Gamma_{n-3}^\ell) + \ell. \tag{6}$$

Here we need the number of edges of Γ_n^k which is obtained in [3] for $n < n_0$ and in [1] for $n \geq n_0$ as follows.

Corollary 2. *[3, 1] The number of edges of Γ_n^k is given by*

$$m(\Gamma_n^k) = \begin{cases} \frac{1}{5}\big(2(n+1)f_n + nf_{n+1}\big) & \text{for } n < n_0 \\ \frac{1}{5}\Big(n_0 f_{n_0-1}L_{t+1} + (n_0-1)f_{n_0}L_{t+2}\Big) + (f_{t+3}-1)k & \text{for } n \geq n_0 \end{cases}$$

where $t = n - n_0$.

By using (6), the number of edges of Γ_n^k in Corollary 2 and the classical identity $L_n = f_{n+1} + f_{n-1}$ we obtain the following result.

Proposition 1. *For $n \geq n_0 = n_0(k)$ the number of edges $m(\Lambda_n^k)$ of Λ_n^k is given by*

- $m(\Lambda_n^k) = (n_0-1)f_{n_0-1} + \ell$ *if* $n = n_0$
- $m(\Lambda_n^k) = \frac{1}{5}\Big(n_0 f_{n_0-1}L_t + (n_0-1)f_{n_0}L_{t+1} + (n-3)L_{n-2} + 2f_{n-3}\Big) + (f_{t+2}-1)k + \ell$ *if* $n_0 + 1 \leq n < n_0(l) + 3$
- $m(\Lambda_n^k) = \frac{1}{5}\Big(n_0 f_{n_0-1}L_t + (n_0-1)f_{n_0}L_{t+1}\Big) + (f_{t+2}-1)k + \frac{1}{5}\Big(n_0(l)f_{n_0(l)-1}L_{t_\ell-2} + (n_0(l)-1)f_{n_0(l)}L_{t_\ell-1}\Big) + f_{t_\ell}\ell$ *if* $n \geq n_0(l) + 3$

where $t = n - n_0$ and $t_\ell = n - n_0(\ell)$.

4.2 The average degree of a vertex

In [9] the limit average degree of the Fibonacci and Lucas cubes are computed as

$$\lim_{n\to\infty} \frac{2m(\Gamma_n)}{nf_{n+2}} = \lim_{n\to\infty} \frac{2m(\Lambda_n)}{nL_n} = 1 - \frac{1}{\sqrt{5}}$$

which means that the average degree of a vertex in Γ_n and Λ_n is asymptotically given by

$$\left(1 - \frac{1}{\sqrt{5}}\right) n \ . \tag{7}$$

The analogous problem for the k-Fibonacci cubes Γ_n^k for a fixed k was considered in [1] where it was proved that the limit average degree $\overline{\deg}(\Gamma_n^k)$ of a vertex in Γ_n^k is independent of n. Denoting this limit average degree by $\overline{d_k}$, we have

$$\overline{d_k} = \frac{1}{5}\left(3 + \sqrt{5}\right) + \left(1 - \frac{1}{\sqrt{5}}\right) \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2}\right) \tag{8}$$

where ϕ is the golden ratio. For the limit average degree of k-Lucas cubes we obtain the following result.

Proposition 2. *For a fixed k the average degree of a vertex in Λ_n^k is asymptotically given by*

$$1.047 + 0.4 \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2}\right) + 0.153 \log_\phi \left(\sqrt{5}\ell + \sqrt{5} - \frac{1}{2}\right)$$

where ϕ is the golden ratio and ℓ is as in (4).

Proof. By the properties of the Fibonacci and Lucas numbers we have

$$\lim_{n\to\infty} \frac{f_{n+1}}{L_n} = \frac{\phi}{\sqrt{5}} \ , \qquad \lim_{n\to\infty} \frac{f_{n-1}}{L_n} = \frac{\phi^{-1}}{\sqrt{5}} \ . \tag{9}$$

For a fixed k, using (6), (8) and (9), the average degree of a vertex in Λ_n^k is computed as

$$\begin{aligned}
\lim_{n\to\infty} \frac{2m(\Lambda_n^k)}{L_n} &= \lim_{n\to\infty} \frac{2\left(m(\Gamma_{n-1}^k) + m(\Gamma_{n-3}^\ell) + \ell\right)}{L_n} \\
&= \lim_{n\to\infty} \frac{2m(\Gamma_{n-1}^k)}{f_{n+1}} \cdot \frac{f_{n+1}}{L_n} + \lim_{n\to\infty} \frac{2m(\Gamma_{n-3}^\ell)}{f_{n-1}} \cdot \frac{f_{n-1}}{L_n} \\
&= \overline{d_k} \cdot \frac{\phi}{\sqrt{5}} + \overline{d_\ell} \cdot \frac{\phi^{-1}}{\sqrt{5}}.
\end{aligned}$$

Using the expressions for $\overline{d_k}$ and $\overline{d_\ell}$ from (8) and simplifying with Mathematica gives the desired result. □

Remark

We note that ℓ is a function of k and using the explicit expression in (4), for large k we obtain the asymptotic value for the average degree in Λ_n^k as

$$\left(1 - \frac{1}{\sqrt{5}}\right) \log_\phi \left(\sqrt{5}k + \sqrt{5} - \frac{1}{2}\right).$$

This is the main term that appears in (8). The factor $1 - \frac{1}{\sqrt{5}}$ is also the coefficient of the limiting values for the Fibonacci and Lucas cubes as given in (7).

4.3 Number of induced hypercubes

Let $Q_d(G)$ denote the number of d-dimensional hypercubes induced in G. This number is considered in the cube polynomials of Fibonacci and Lucas cubes in [8]. For k-Fibonacci cubes, it is shown in [1] that $Q_d(\Gamma_n^k)$ satisfies the recursion

$$Q_d(\Gamma_n^k) = Q_d(\Gamma_{n-1}^k) + Q_d(\Gamma_{n-2}^k) + P_{d-1}(k-1) \tag{10}$$

where

$$P_{d-1}(k-1) = \sum_{i=0}^{k-1} \binom{Z(i)}{d-1},$$

and $Z(i)$ denotes the number of 1's in the Zeckendorf representation of i. The idea behind the proof of (10) is as follows. The first and the second term on the right hand side of the equation (10) follow immediately from the fundamental decomposition $\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k$. The term $P_{d-1}(k-1)$ counts the number of hypercubes that involve the k link edges used in the construction of Γ_n^k. In [1], these hypercubes are counted by the number of $(d-1)$-dimensional hypercubes contained in the subgraph of $0\Gamma_{n-1}^k$ induced by the first k vertices with labels $0, 1, \ldots, k-1$. The claim is that for any of these vertices i, the selection of $d-1$ ones among the $Z(i)$ ones in the expansion of i gives one $d-1$ dimensional hypercube, since by varying any of these $d-1$ ones we get 2^{d-1} vertices with labels in $\{0, 1, \ldots, k-1\}$ each giving a $(d-1)$-dimensional hypercube in $0\Gamma_{n-1}^k$. Furthermore, there is a copy of this hypercube in $10\Gamma_{n-2}^k$ and also a matching between these hypercubes due to the k link edges, producing a d-dimensional hypercube in Γ_n^k.

For k-Lucas cubes, we use a similar argument to find the number of d-dimensional hypercubes induced in Λ_n^k. From Theorem 1 we know that $\Lambda_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-3}^\ell 0$. Therefore, there are three types of d-dimensional hypercubes that contribute to $Q_d(\Lambda_n^k)$: those coming from $0\Gamma_{n-1}^k$, those coming from $10\Gamma_{n-3}^\ell 0$, and those that involve the ℓ link edges used in the construction of Λ_n^k. It is enough to consider the d-dimensional hypercubes of the last type. These can be counted by the number of $(d-1)$-dimensional hypercubes contained in the subgraph of $10\Gamma^{\ell}_{n-3}0$ induced by the ℓ vertices with labels in $\{0, 1, \ldots, k-1\}$ having "even" Zeckendorf expansions, that is, whose representations that end with 0. For any of these vertices i again we need to select $d-1$ ones among the $Z(i)$ ones in i. Then by varying these $d-1$ ones we obtain 2^{d-1} vertices with labels in $\{0, 1, \ldots, k-1\}$ having even Zeckendorf expansions themselves. Each one of these gives a $(d-1)$-dimensional hypercube in $10\Gamma^{\ell}_{n-3}0$. All of these $(d-1)$-dimensional hypercubes also have a copy in $0\Gamma^{k}_{n-1}$ and there is a matching between the two hypercubes due to the ℓ link edges. This produces a d-dimensional hypercube in Λ^k_n that involves the link edges. We have the following result:

Proposition 3. *Let $Q_d(\Lambda^k_n)$ and $Q_d(\Gamma^k_n)$ denote the number of d-dimensional hypercubes in Λ^k_n and Γ^k_n respectively. Then*

$$Q_d(\Lambda^k_n) = Q_d(\Gamma^k_{n-1}) + Q_d(\Gamma^{\ell}_{n-3}) + P_{d-1}(\ell - 1).$$

Proof. The bulk of the proof of the proposition has been given above, showing

$$Q_d(\Lambda^k_n) = Q_d(\Gamma^k_{n-1}) + Q_d(\Gamma^{\ell}_{n-3}) + \sum_{i \in S} \binom{Z(i)}{d-1},$$

where S is the ℓ integers in $\{0, 1, \ldots, k-1\}$ having even Zeckendorf expansions. To show that

$$\sum_{i \in S} \binom{Z(i)}{d-1} = \sum_{i=0}^{\ell-1} \binom{Z(i)}{d-1} = P_{d-1}(\ell - 1) \tag{11}$$

we argue as follows. The Zeckendorf expansions of the numbers $\{0, 1, \ldots k-1\}$ can be partitioned into the disjoint union of two sets of expansions of the form $A{\cdot}0$ and $B{\cdot}01$ where A is the Zeckendorf expansion of the numbers $\{0, 1, \ldots, \ell-1\}$ and B is the Zeckendorf expansion of the numbers $\{0, 1, \ldots, \lfloor \frac{k+1}{\phi^2} \rfloor - 1\}$. Since the number of ones of the even Zeckendorf numbers in $\{0, 1, \ldots, k-1\}$ does not change when we drop the last 0, the sums in (11) are identical. □

4.4 Diameter and radius

Γ^k_n has the nested structure

$$\Gamma^1_n \subseteq \cdots \subseteq \Gamma^k_n \subseteq \cdots \subseteq \Gamma_n.$$

as shown in [1]. Since we define Λ^k_n by removing certain vertices in Γ^k_n, one can easily observe that k-Lucas cubes have a similar nested structure,

$$\Lambda^1_n \subseteq \cdots \subseteq \Lambda^k_n \subseteq \cdots \subseteq \Lambda_n. \tag{12}$$

We know that Λ^1_n is a tree with root 0^n (the vertex with integer label 0). It follows that for $u, v \in V(\Lambda^1_n)$

$$d(u, v) \le d(u, 0^n) + d(v, 0^n) = w_H(u) + w_H(v), \tag{13}$$

where w_H denotes the Hamming weight. We always have

$$w_H(u) + w_H(v) \leq \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd} \end{cases}$$

for the vertices of Λ_n and it is shown in [12] that

$$diam(\Lambda_n) = \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd.} \end{cases}$$

Λ_n^k is a subgraph of Λ_n with the same vertex set and fewer edges for $n \geq n_0$. This directly gives the inequality $diam(\Lambda_n^k) \geq diam(\Lambda_n)$. On the other hand, using (12) and (13), for any $u, v \in V(\Lambda_n^k)$ we have

$$d(u,v) \leq w_H(u) + w_H(v) \leq \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd,} \end{cases}$$

which gives $diam(\Lambda_n^k) \leq diam(\Lambda_n)$. Therefore, for all $n \geq 1$

$$diam(\Lambda_n^k) = diam(\Lambda_n) = \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd.} \end{cases}$$

By a similar argument we see that the radius of Λ_n^k is equal to the radius of Λ_n. Since the latter radius was obtained in [12] as $\lfloor \frac{n}{2} \rfloor$, this is also the radius of Λ_n^k.

Acknowledgments

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A Figures of some k-Lucas cubes

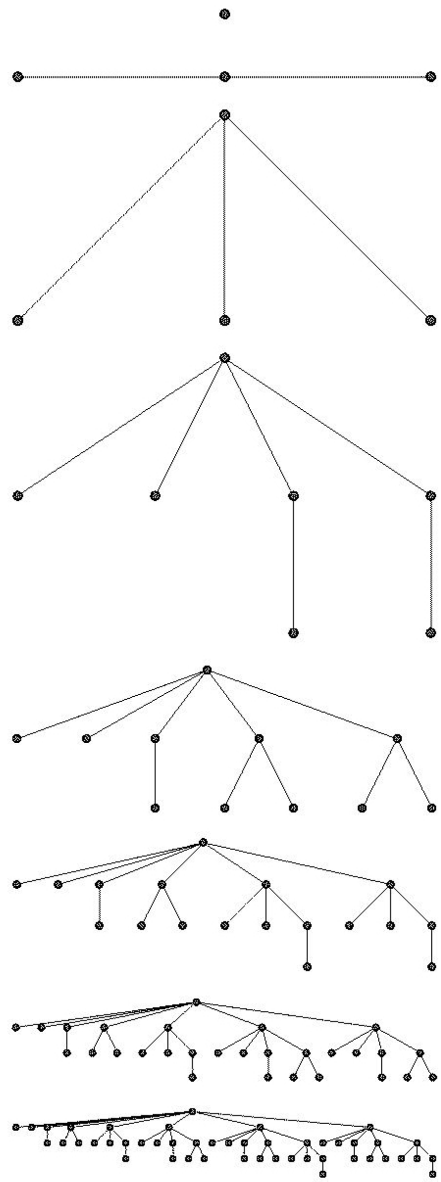

Figure 4: The first eight k-Lucas cubes for $k = 1$.

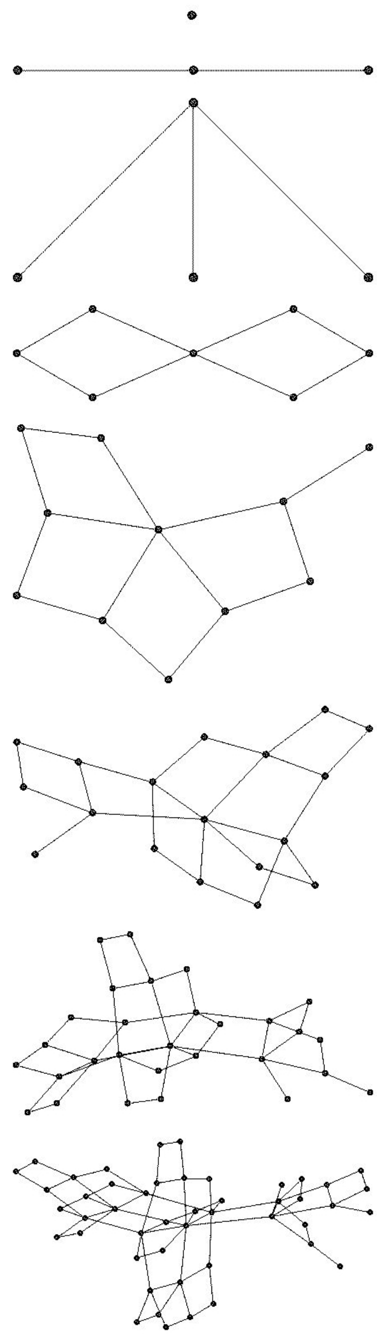

Figure 5: The first eight k-Lucas cubes for $k = 3$.

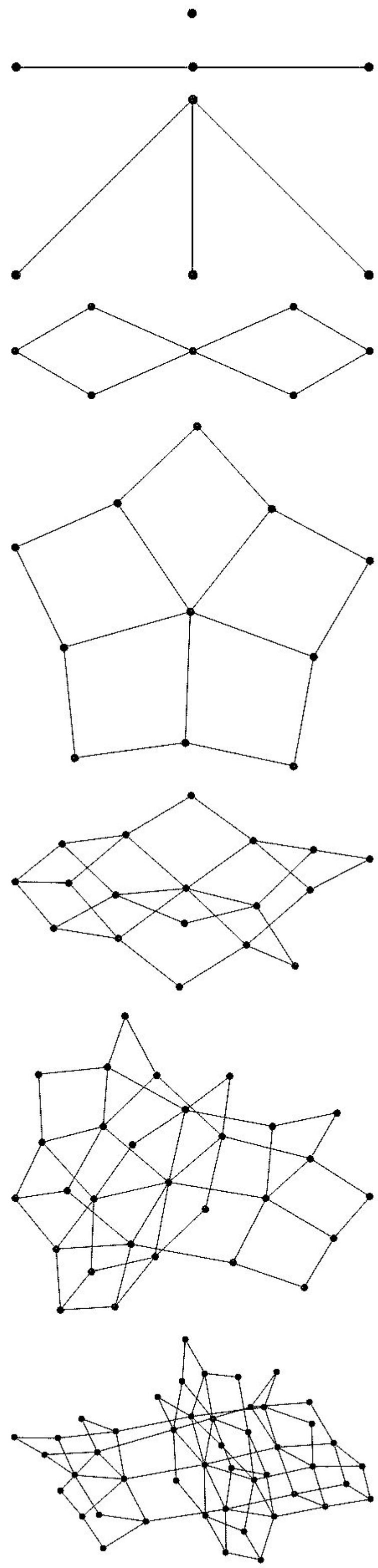

Figure 6: The first eight k-Lucas cubes for $k = 6$.

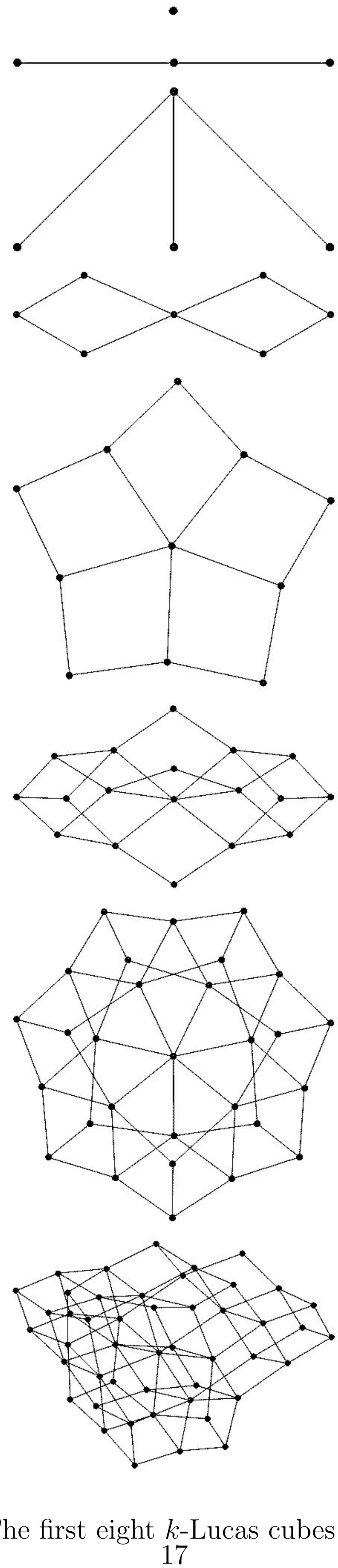

Figure 7: The first eight k-Lucas cubes for $k = 12$.