13. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.

14. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

15. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

17. Suppose that $A$, $B$, and $C$ are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

18. Find two sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$.

19. What is the cardinality of each of these sets?
   a) $\{a\}$
   b) $\{\{a\}\}$
   c) $\{a, \{a\}\}$
   d) $\{a, \{a\}, \{a, \{a\}\}\}$

20. What is the cardinality of each of these sets?
   a) $\emptyset$
   b) $\{\emptyset\}$
   c) $\{\emptyset, \{\emptyset\}\}$
   d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

21. Find the power set of each of these sets, where $a$ and $b$ are distinct elements.
   a) $\{a\}$
   b) $\{a, b\}$
   c) $\{\emptyset, \{\emptyset\}\}$

22. Can you conclude that $A = B$ if $A$ and $B$ are two sets with the same power set?

23. How many elements does each of these sets have where $a$ and $b$ are distinct elements?
   a) $P(\{a, b, \{a, b\}\})$
   b) $P(\emptyset, a, \{a\})$
   c) $P(\emptyset)$

24. Determine whether each of these sets is the power set of a set, where $a$ and $b$ are distinct elements.
   a) $\emptyset$
   b) $\emptyset, \{a\}$
   c) $\emptyset, \{a\}, \{\emptyset, a\}$
   d) $\emptyset, \{a\}, \{b\}, \{a, b\}$

25. Prove that $P(A) \subseteq P(B)$ if and only if $A \subseteq B$.

26. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

27. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find
   a) $A \times B$
   b) $B \times A$.

28. What is the Cartesian product $A \times B$, where $A$ is the set of courses offered by the mathematics department at a university and $B$ is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

29. What is the Cartesian product $A \times B \times C$, where $A$ is the set of all airlines and $B$ and $C$ are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

30. Suppose that $A \times B = \emptyset$, where $A$ and $B$ are sets. What can you conclude?

31. Let $A$ be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

32. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find
   a) $A \times B \times C$.
   b) $C \times B \times A$.
   c) $C \times A \times B$.
   d) $B \times B \times B$.

33. Find $A^2$ if
   a) $A = \{0, 1, 3\}$.
   b) $A = \{1, 2, a, b\}$.

34. Find $A^3$ if
   a) $A = \{a\}$.
   b) $A = \{0, a\}$.

35. How many different elements does $A \times B$ have if $A$ has $m$ elements and $B$ has $n$ elements?

36. How many different elements does $A \times B \times C$ have if $A$ has $m$ elements, $B$ has $n$ elements, and $C$ has $p$ elements?

37. How many different elements does $A^n$ have when $A$ has $m$ elements and $n$ is a positive integer?

38. Show that $A \times B \neq B \times A$, when $A$ and $B$ are nonempty, unless $A = B$.

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

40. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

41. Translate each of these quantifications into English and determine its truth value.
   a) $\forall x \in R \ (x^2 \neq -1)$
   b) $\exists x \in Z \ (x^2 = 2)$
   c) $\forall x \in Z \ (x^2 > 0)$
   d) $\exists x \in R \ (x^2 = x)$

42. Translate each of these quantifications into English and determine its truth value.
   a) $\exists x \in R \ (x^3 = -1)$
   b) $\exists x \in Z \ (x + 1 > x)$
   c) $\forall x \in Z \ (x - 1 \in Z)$
   d) $\forall x \in Z \ (x^2 \in Z)$

43. Find the truth set of each of these predicates where the domain is the set of integers.
   a) $P(x): x^2 < 3$
   b) $Q(x): x^2 > x$
   c) $R(x): 2x + 1 = 0$

44. Find the truth set of each of these predicates where the domain is the set of integers.
   a) $P(x): x^3 \geq 1$
   b) $Q(x): x^2 = 2$
   c) $R(x): x < x^2$

45. The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair $(a, b)$ to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [Hint: First show that $\{\{a\}, \{a, b\}\} = \{\{a\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

46. This exercise presents Russell’s paradox. Let $S$ be the set that contains a set $x$ if the set $x$ does not belong to itself, so that $S = \{x \mid x \notin x\}$.
   a) Show the assumption that $S$ is a member of $S$ leads to a contradiction.
   b) Show the assumption that $S$ is not a member of $S$ leads to a contradiction.

   By parts (a) and (b) it follows that the set $S$ cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

47. Describe a procedure for listing all the subsets of a finite set.
30. Can you conclude that \( A = B \) if \( A, B, \) and \( C \) are sets such that
   a) \( A \cup C = B \cup C \)
   b) \( A \cap C = B \cap C \)
   c) \( A \cup C = B \cup C \) and \( A \cap C = B \cap C \)
31. Let \( A \) and \( B \) be subsets of a universal set \( U \). Show that \( A \subseteq B \) if and only if \( B \subseteq A \).

   The **symmetric difference** of \( A \) and \( B \), denoted by \( A \oplus B \), is the set containing those elements in either \( A \) or \( B \), but not in both \( A \) and \( B \).
32. Find the symmetric difference of \( \{1, 3, 5\} \) and \( \{1, 2, 3\} \).
33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.
34. Draw a Venn diagram for the symmetric difference of the sets \( A \) and \( B \).
35. Show that \( A \oplus B = (A \cup B) - (A \cap B) \).
36. Show that \( A \oplus B = (A - B) \cup (B - A) \).
37. Show that if \( A \) is a subset of a universal set \( U \), then
   a) \( A \oplus \emptyset = A \)
   b) \( A \oplus U = U \)
   c) \( A \oplus A = \emptyset \)
   d) \( A \oplus U = U \)
38. Show that if \( A \) and \( B \) are sets, then
   a) \( A \oplus B = B \oplus A \)
   b) \( (A \oplus B) \oplus C = A \oplus (B \oplus C) \)
39. What can you say about the sets \( A \) and \( B \) if \( A \oplus B = A \)?

   *40. Determine whether the symmetric difference is associative; that is, if \( A, B, \) and \( C \) are sets, does it follow that \( A \oplus (B \oplus C) = (A \oplus B) \oplus C \)?

   *41. Suppose that \( A, B, \) and \( C \) are sets such that \( A \oplus C = B \oplus C \). Must it be the case that \( A = B \)?
42. If \( A, B, C, \) and \( D \) are sets, does it follow that \( (A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D) \)?
43. If \( A, B, C, \) and \( D \) are sets, does it follow that \( (A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C) \)?
44. Show that if \( A \) and \( B \) are finite sets, then \( A \cup B \) is a finite set.
45. Show that if \( A \) is an infinite set, then whenever \( B \) is a set, \( A \cup B \) is also an infinite set.

   *46. Show that if \( A, B, \) and \( C \) are sets, then
   \[
   |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.
   \]

   (This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

47. Let \( A_i = \{1, 2, 3, \ldots, i\} \) for \( i = 1, 2, 3, \ldots \).
   a) \( \bigcup_{i=1}^{n} A_i \)
   b) \( \bigcap_{i=1}^{n} A_i \)
48. Let \( A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\} \).
   a) \( \bigcup_{i=1}^{n} A_i \)
   b) \( \bigcap_{i=1}^{n} A_i \)
49. Let \( A_i \) be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding \( i \).
   a) \( \bigcup_{i=1}^{n} A_i \)
   b) \( \bigcap_{i=1}^{n} A_i \)

50. Find \( \bigcup_{i=1}^{n} A_i \) and \( \bigcap_{i=1}^{n} A_i \) if for every positive integer \( i \),
   a) \( A_i = \{i, i+1, i+2,\ldots\} \)
   b) \( A_i = \{0, i\} \)
   c) \( A_i = (0, i) \) that is, the set of real numbers \( x \) with \( 0 < x < i \).
   d) \( A_i = (i, \infty) \) that is, the set of real numbers \( x \) with \( x > i \).

51. Find \( \bigcup_{i=1}^{n} A_i \) and \( \bigcap_{i=1}^{n} A_i \) if for every positive integer \( i \),
   a) \( A_i = \{-i, -i+1, \ldots, -1, 0, 1, \ldots, i-1, i\} \)
   b) \( A_i = \{-i, i\} \)
   c) \( A_i = [-i, i] \) that is, the set of real numbers \( x \) with \( -i \leq x \leq i \).
   d) \( A_i = [i, \infty) \) that is, the set of real numbers \( x \) with \( x \geq i \).

52. Suppose that the universal set is \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Express each of these sets with bit strings where the \( i \)th bit in the string is 1 if \( i \) is in the set and 0 otherwise.
   a) \( \{3, 4, 5\} \)
   b) \( \{1, 3, 6, 10\} \)
   c) \( \{2, 3, 4, 7, 8, 9\} \)

53. Using the same universal set as in the last problem, find the set specified by each of these bit strings.
   a) \( 11 \ 1100 \ 1111 \)
   b) \( 01 \ 0111 \ 1000 \)
   c) \( 10 \ 0000 \ 0001 \)

54. What subsets of a finite universal set do these bit strings represent?
   a) the string with all zeros
   b) the string with all ones

55. What is the bit string corresponding to the difference of two sets?

56. What is the bit string corresponding to the symmetric difference of two sets?

57. Show how bitwise operations on bit strings can be used to find these combinations of \( A = \{a, b, c, d, e\} \),
   \( B = \{b, c, d, g, p, t, v\} \), \( C = \{c, e, i, o, u, x, y, z\} \), and \( D = \{d, e, h, i, n, o, t, u, x, y\} \).
   a) \( A \cup B \)
   b) \( A \cap B \)
   c) \( (A \cup D) \cap (B \cup C) \)
   d) \( A \cup B \cup C \cup D \)

58. How can the union and intersection of \( n \) sets that all are subsets of the universal set \( U \) be found using bit strings?

   The **successor** of the set \( A \) is the set \( A \cup \{A\} \).
59. Find the successors of the following sets.
   a) \( \{1, 2, 3\} \)
   b) \( \emptyset \)
   c) \( \{\emptyset\} \)
   d) \( \{\emptyset, \{\emptyset\}\} \)
59. How many bytes are required to encode \( n \) bits of data where \( n \) equals
   a) 7?  b) 17?  c) 1001?  d) 28,800?

60. How many ATM cells (described in Example 28) can be transmitted in 10 seconds over a link operating at the following rates?
   a) 128 kilobits per second (1 kilobit = 1000 bits)
   b) 300 kilobits per second
   c) 1 megabit per second (1 megabit = 1,000,000 bits)

61. Data are transmitted over a particular Ethernet network in blocks of 1500 octets (blocks of 8 bits). How many blocks are required to transmit the following amounts of data over this Ethernet network? (Note that a byte is a synonym for an octet, a kilobyte is 1000 bytes, and a megabyte is 1,000,000 bytes.)
   a) 150 kilobytes of data
   b) 384 kilobytes of data
   c) 1.544 megabytes of data
   d) 45.3 megabytes of data

62. Draw the graph of the function \( f(n) = 1 - n^2 \) from \( Z \) to \( Z \).

63. Draw the graph of the function \( f(x) = [2x] \) from \( R \) to \( R \).

64. Draw the graph of the function \( f(x) = [x/2] \) from \( R \) to \( R \).

65. Draw the graph of the function \( f(x) = [x] + [x/2] \) from \( R \) to \( R \).

66. Draw the graph of the function \( f(x) = [x] + [x/2] \) from \( R \) to \( R \).

67. Draw graphs of each of these functions.
   a) \( f(x) = [x + \frac{1}{2}] \)
   b) \( f(x) = [2x + 1] \)
   c) \( f(x) = \frac{[x]}{3} \)
   d) \( f(x) = [1/x] \)
   e) \( f(x) = [x - 2] + [x + 2] \)
   f) \( f(x) = [2x] [x/2] \)
   g) \( f(x) = [x - \frac{1}{2}] + \frac{1}{2} \)

68. Draw graphs of each of these functions.
   a) \( f(x) = [3x - 2] \)
   b) \( f(x) = [0.2x] \)
   c) \( f(x) = [-1/x] \)
   d) \( f(x) = [x^2] \)
   e) \( f(x) = [x/2] [x/2] \)
   f) \( f(x) = [x/2] + [x/2] \)
   g) \( f(x) = [2x/2] + \frac{1}{2} \)

69. Find the inverse function of \( f(x) = x^3 + 1 \).

70. Suppose that \( f \) is an invertible function from \( Y \) to \( Z \) and \( g \) is an invertible function from \( X \) to \( Y \). Show that the inverse of the composition \( f \circ g \) is given by \((f \circ g)^{-1} = g^{-1} \circ f^{-1}\). Show that if \( f \) is an invertible function from \( X \) to \( Y \), then \( f^{-1} \) is the inverse of \( f \).

71. Let \( S \) be a subset of a universal set \( U \). The characteristic function \( f_S \) of \( S \) is the function from \( U \) to \( \{0, 1\} \) such that \( f_S(x) = 1 \) if \( x \) belongs to \( S \) and \( f_S(x) = 0 \) if \( x \) does not belong to \( S \). Let \( A \) and \( B \) be sets. Show that for all \( x \in U \),
   a) \( f_{A \cap B}(x) = f_A(x) \cdot f_B(x) \)
   b) \( f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) \)
   c) \( f_{\overline{A}}(x) = 1 - f_A(x) \)
   d) \( f_{A \setminus B}(x) = f_A(x) + f_B(x) - 2f_A(x) f_B(x) \)

\textbf{72.}\: Suppose that \( f \) is a function from \( A \) to \( B \), where \( A \) and \( B \) are finite sets with \(|A| = |B|\). Show that \( f \) is one-to-one if and only if it is onto.

73. Prove or disprove each of these statements about the floor and ceiling functions.
   a) \( \lfloor x \rfloor = \lfloor y \rfloor \) for all real numbers \( x \).
   b) \( \lfloor 2x \rfloor = 2 \lfloor x \rfloor \) whenever \( x \) is a real number.
   c) \( \lfloor x \rfloor + \lfloor y \rfloor - \lfloor x + y \rfloor = 0 \) or 1 whenever \( x \) and \( y \) are real numbers.
   d) \( \lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor \) for all real numbers \( x \) and \( y \).
   e) \( \lfloor x/2 \rfloor = \lfloor x+1/2 \rfloor \) for all real numbers \( x \).

74. Prove or disprove each of these statements about the floor and ceiling functions.
   a) \( \lfloor x \rfloor = \lfloor y \rfloor \) for all real numbers \( x \).
   b) \( \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \) for all real numbers \( x \) and \( y \).
   c) \( \lfloor x/2 \rfloor / \lfloor 2 \rfloor = \lfloor x/4 \rfloor \) for all real numbers \( x \).
   d) \( \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \) for all positive real numbers \( x \).
   e) \( \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor \) for all real numbers \( x \) and \( y \).

75. Prove that if \( x \) is a positive real number, then
   a) \( \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \).
   b) \( \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \).

76. Let \( x \) be a real number. Show that \( \lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor \).

77. For each of these partial functions, determine its domain, codomain, domain of definition, and the set of values for which it is undefined. Also, determine whether it is a total function.
   a) \( f : Z \rightarrow R, f(n) = 1/n \)
   b) \( f : Z \rightarrow Z, f(n) = \lfloor n/2 \rfloor \)
   c) \( f : Z \times Z \rightarrow Q, f(m, n) = m/n \)
   d) \( f : Z \times Z \rightarrow Z, f(m, n) = mn \)
   e) \( f : Z \times Z \rightarrow Z, f(m, n) = m - n \) if \( m > n \)

78. a) Show that a partial function from \( A \) to \( B \) can be viewed as a function \( f^* \) from \( A \) to \( B \cup \{u\} \), where \( u \) is not an element of \( B \) and

   \[
   f^*(a) = \begin{cases} 
   f(a) & \text{if } a \text{ belongs to the domain of definition of } f \\
   u & \text{if } f \text{ is undefined at } a.
   \end{cases}
   \]

b) Using the construction in (a), find the function \( f^* \) corresponding to each partial function in Exercise 77.

\textbf{79.}\: a) Show that if a set \( S \) has cardinality \( m \), where \( m \) is a positive integer, then there is a one-to-one correspondence between \( S \) and the set \( \{1, 2, \ldots, m\} \).

b) Show that if \( S \) and \( T \) are two sets each with \( m \) elements, where \( m \) is a positive integer, then there is a one-to-one correspondence between \( S \) and \( T \).

\textbf{80.}\: Show that a set \( S \) is infinite if and only if there is a proper subset \( A \) of \( S \) such that there is a one-to-one correspondence between \( A \) and \( S \).
23. Find a recurrence relation for the balance $B(k)$ owed at the end of $k$ months on a loan of $5000 at a rate of 7% if a payment of $100 is made each month. [Hint: Express $B(k)$ in terms of $B(k-1)$; the monthly interest is $0.07/12B(k-1).$]

24. a) Find a recurrence relation for the balance $B(k)$ owed at the end of $k$ months on a loan at a rate of $r$ if a payment $P$ is made on the loan each month. [Hint: Express $B(k)$ in terms of $B(k-1)$ and note that the monthly interest rate is $r/12.$]

b) Determine what the monthly payment $P$ should be so that the loan is paid off after $T$ months.

25. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) $1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, \ldots$

b) $1, 2, 3, 4, 5, 6, 7, 8, 8, \ldots$

c) $1, 0, 2, 0, 4, 0, 8, 0, 16, 0, \ldots$

d) $3, 6, 12, 24, 48, 96, 192, \ldots$

e) $15, 8, 1, -6, -13, -20, -27, \ldots$

f) $3, 5, 8, 12, 17, 23, 30, 38, 47, \ldots$

g) $2, 16, 54, 128, 250, 432, 686, \ldots$

h) $2, 3, 7, 25, 121, 721, 5041, 40321, \ldots$

26. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) $3, 6, 11, 18, 27, 38, 51, 66, 83, 102, \ldots$

b) $7, 11, 15, 19, 23, 27, 31, 35, 39, 43, \ldots$

c) $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, \ldots$

d) $1, 2, 2, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, \ldots$

27. Show that if $a_n$ denotes the $n$th positive integer that is not a perfect square, then $a_n = n + \lfloor \sqrt{n} \rfloor$, where $\lfloor x \rfloor$ denotes the integer closest to the real number $x$.

28. Let $a_n$ be the $n$th term of the sequence $1, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ldots$, constructed by including the integer $k$ exactly $k$ times. Show that $a_n = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$.

29. What are the values of these sums?

a) $\sum_{k=1}^{5} (k + 1)$

b) $\sum_{j=0}^{4} (-2)^j$

c) $\sum_{i=1}^{10} 3$

d) $\sum_{j=0}^{8} (2j+1 - 2^j)$

30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

a) $\sum_{j \in S} j$

b) $\sum_{j \in S} j^2$

c) $\sum_{j \in S} (1/j)$

d) $\sum_{j \in S} 1$

31. What is the value of each of these sums of terms of a geometric progression?

a) $\sum_{j=0}^{8} 3 \cdot 2^j$

b) $\sum_{j=0}^{8} 2^j$

c) $\sum_{j=2}^{8} (-3)^j$

d) $\sum_{j=0}^{8} 2 \cdot (-3)^j$

32. Find the value of each of these sums.

a) $\sum_{j=0}^{8} (1 + (-1)^j)$

b) $\sum_{j=0}^{8} (3j - 2j)$

c) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$

d) $\sum_{j=0}^{8} (2j+1 - 2^j)$

33. Compute each of these double sums.

a) $\sum_{i=1}^{3} \sum_{j=1}^{3} (i + j)$

b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i + 3j)$

c) $\sum_{i=1}^{3} \sum_{j=1}^{3} ij$

d) $\sum_{i=0}^{3} \sum_{j=0}^{3} i^2 j^3$

34. Compute each of these double sums.

a) $\sum_{i=1}^{3} \sum_{j=1}^{3} (i - j)$

b) $\sum_{i=0}^{2} \sum_{j=0}^{3} (3i + 2j)$

c) $\sum_{i=1}^{3} \sum_{j=1}^{3} j$

d) $\sum_{i=0}^{3} \sum_{j=0}^{3} i^2 j^3$

35. Show that $\sum_{j=m}^{n} (a_j - a_{j-1}) = a_n - a_m$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called telescoping.

36. Use the identity $1/(k(k+1)) = 1/k - 1/(k+1)$ and Exercise 35 to compute $\sum_{k=1}^{n} 1/(k(k+1))$.

37. Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use Exercise 35 to find

a) a formula for $\sum_{k=1}^{n} (2k - 1)$ (the sum of the first $n$ odd natural numbers).

b) a formula for $\sum_{k=1}^{n} k$.

*38. Use the technique given in Exercise 35, together with the result of Exercise 37b, to derive the formula for $\sum_{k=1}^{n} k^2$ given in Table 2. [Hint: Take $a_k = k^3$ in the telescoping sum in Exercise 35.]

39. Find $\sum_{k=1}^{200} k^2$. (Use Table 2.)

40. Find $\sum_{k=1}^{200} k^3$. (Use Table 2.)

*41. Find a formula for $\sum_{k=0}^{m} [\sqrt{k}]$, when $m$ is a positive integer.

*42. Find a formula for $\sum_{k=0}^{m} [\sqrt{k}]$, when $m$ is a positive integer.

There is also a special notation for products. The product of $a_0, a_m+1, \ldots, a_n$ is represented by $\prod_{j=m}^{n} a_j$, read as the product from $j = m$ to $j = n$ of $a_j$. 
Exercises

1. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   a) the negative integers
   b) the even integers
   c) the integers less than 100
   d) the real numbers between 0 and 1
   e) the positive integers less than 1,000,000,000
   f) the integers that are multiples of 7

2. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   a) the integers greater than 10
   b) the odd negative integers
   c) the integers with absolute value less than 1,000,000
   d) the real numbers between 0 and 2
   e) the set \( \mathbb{A} \times \mathbb{Z}^+ \) where \( \mathbb{A} = \{2, 3\} \)
   f) the integers that are multiples of 10

3. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   a) all bit strings not containing the bit 0
   b) all positive rational numbers that cannot be written with denominators less than 4
   c) the real numbers not containing 0 in their decimal representation
   d) the real numbers containing only a finite number of 1s in their decimal representation

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   a) integers not divisible by 3
   b) integers divisible by 5 but not by 7
   c) the real numbers with decimal representations consisting of all 1s or 9s
   d) the real numbers with decimal representations of all 1s or 9s

5. Show that a finite group of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

6. Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Show that all guests can remain in the hotel.

7. Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.

8. Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

9. Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Show that all the arriving guests can be accommodated without evicting any current guest.

10. Give an example of two uncountable sets \( A \) and \( B \) such that \( A - B \) is
   a) finite.
   b) countably infinite.
   c) uncountable.

11. Give an example of two uncountable sets \( A \) and \( B \) such that \( A \cap B \) is
   a) finite.
   b) countably infinite.
   c) uncountable.

12. Show that if \( A \) and \( B \) are sets and \( A \subset B \) then \( |A| \leq |B| \).

13. Explain why the set \( A \) is countable if and only if \( |A| \leq |\mathbb{Z}^+| \).

14. Show that if \( A \) and \( B \) are sets with the same cardinality, then \( |A| \leq |B| \) and \( |B| \leq |A| \).

15. Show that if \( A \) and \( B \) are sets, \( A \) is uncountable, and \( A \subseteq B \), then \( B \) is uncountable.

16. Show that a subset of a countable set is also countable.

17. If \( A \) is an uncountable set and \( B \) is a countable set, must \( A - B \) be uncountable?