Frequency Domain Methods for Optimal Periodic Control

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Example: CSTR with two second order reactions

\[ \dot{x}_1 = -ux_1^\alpha - au^\theta x_1 - x_1 + 1 \]

\[ \dot{x}_2 = ux_1^\alpha - x_2 \]

\[ q, A \quad q, A, B, C \]

A \rightarrow B
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- Optimal Periodic Control for Thermoacoustics as a particular application.
General Problem Setting

\[
\text{minimize } \quad J(x(0), u(\cdot), T) = \frac{1}{T} \int_0^T L(x(t), u(t)) \, dt
\]

subject to

\[
\begin{align*}
\dot{x} &= f(x, u) \\
x(0) &= x(T) \\
\int_0^T v(x, u) \, dt &= 0 \\
\int_0^T w(x, u) \, dt &\leq 0
\end{align*}
\]
Particular Setting

\[
J(x(0), u(\cdot), T) = \min_{x(0), u(\cdot) \in \mathcal{U}, T} \frac{1}{T} \int_0^T \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \phi(C x) \, dt
\]

subject to

\[
\begin{aligned}
\dot{x} &= Ax + Bu \\
x(0) &= x(T)
\end{aligned}
\]

- \( \phi(y) = \gamma_1 y + \gamma_2 y^2 + \ldots + \frac{1}{p} y^p \).
- \( x \in \mathbb{R}^n, u \in \mathbb{R}, C \in \mathbb{R}^{1 \times n}, B \in \mathbb{R}^{n \times m}, Q \in \mathbb{R}^{n \times n}, 0 < R \in \mathbb{R}^{m \times m} \).
First order necessary conditions

With the Hamiltonian

\[ H(x, u, \lambda) := \frac{1}{2} (x^T Q x + u^T R u) + \phi(C x) + \lambda^T (A x + B u), \]

we get the Hamiltonian System

\[
\begin{align*}
\dot{x} &= A x + B u \\
\dot{\lambda} &= -A^T \lambda - Q x - C^T \frac{\partial \phi}{\partial y} (C x) = - \frac{\partial H}{\partial x} \\
0 &= R u + B^T \lambda = \frac{\partial H}{\partial u} \\
\lambda(0) &= \lambda(T) \\
x(0) &= x(T) \\
0 &= J(u, x, T) - H(x(T), u(T), \lambda(T)).
\end{align*}
\]

Thus any optimal trajectory corresponds to a \textit{periodic} solution of above system. That is true for optimal periodic control problems in general.
Finding periodic solutions: Harmonic Balance

Every periodic signal can be represented by its Fourier series, thus for any periodic solution $y(t) = \sum_{k \in \mathbb{Z}} \alpha_k e^{jk\omega t}$, we have

$$v(t) = \sum_{k \in \mathbb{Z}} \beta_k(\alpha) e^{jk\omega t},$$

and component-wise

$$\alpha_k - H(jk\omega t)\beta_k(\alpha) = 0 \quad \forall k \in \mathbb{Z}.$$
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A periodic solution needs to self-excite:

\[ \sum_{k \in \mathbb{Z}} \alpha_k e^{jk\omega t} = \sum_{k \in \mathbb{Z}} H(jk\omega t) \beta_k(\alpha) e^{jk\omega t}, \]

and component-wise

\[ \alpha_k - H(jk\omega t) \beta_k(\alpha) = 0 \quad \forall k \in \mathbb{Z}. \]
Homotopy Continuation Methods

Let \( F(x) = 0 \) be system of polynomial equations in \( n \) indeterminates \( x_i \).

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2 Verschelde, Jan “Algorithm 795: PHCPACK: A general-purpose solver for polynomial systems by homotopy continuation,” ACM Transactions on Mathematical Software, 1999
Homotopy Continuation Methods

- Let $F(x) = 0$ be system of polynomial equations in $n$ indeterminates $x_i$.
- The homotopy is a system with an additional parameter $t \in [0, 1]$ which embeds $F(x)$:

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\Phi(x, t) = tF(x) + (1 - t)F_0(x), \quad t \in [0, 1].
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- PHCpack\(^2\) is the software implementation of Homotopy Methods that we use.

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5 Unleash PHCpack on it

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Convert Hamiltonian System to Lur’e structure

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\dot{\lambda} &= -A^T \lambda - Qx - C^T \frac{\partial \phi}{\partial y}(Cx) = -\frac{\partial H}{\partial x} \\
0 &= Ru + B^T \lambda = \frac{\partial H}{\partial u} \\
\lambda(0) &= \lambda(T) \\
x(0) &= x(T)
\end{align*}
\]
1 Convert Hamiltonian System to Lur’e structure

\[ \frac{\partial \phi}{\partial y}(\cdot) = H(s)\psi(y) \]

\[ R^{-1} \]

\[ G(s) \]

\[ C \]

\[ Q \]

\[ -B^T \lambda \]

\[ -G^T(-s) \]

\[ Qx + C^Tv \]

\[ y \]

\[ \psi(y) \]

\[ v \]

\[ \frac{\partial \phi}{\partial y}(\cdot) \]
1 Convert Hamiltonian System to Lur’e structure

\[
H(s) = -C G(s) R^{-1} \left( I + G^T(-s)Q G(s) R^{-1} \right)^{-1} G^T(-s) C^T
\]

\[
\psi(y) = \frac{\partial \phi}{\partial y}(y) = y^{p-1} + \sum_{q=0}^{p-2} (q+1) \gamma_{q+1} y^q
\]
Convert Hamiltonian System to Lur’e structure

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2 Use Harmonic Balance

\[ \forall k : \alpha_k - H(j\omega_k) \beta_k(\alpha) = 0 \]
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\[ \forall k : \quad \alpha_k - H(j\omega k) \beta_k(\alpha) = 0 \]

3 Find closed form of \( \beta_k(\alpha) \)

For a single power we find functions \( \Psi_{k,q}(\cdot) \) such that

\[
\left( \sum_{k=-N}^{N} \alpha_k e^{jk\omega t} \right)^q = \sum_{k=-qN}^{qN} \Psi_{k,q}(\alpha) e^{jk\omega t},
\]

Combinatorics

\[
\Psi_{k,q}(\alpha) = \sum_{\ell_1+\ell_2+\ldots+\ell_q=k}^{\ell_i \in [-N,N]} \frac{q}{c_{-N}, \ldots, c_N} \alpha_{\ell_1} \alpha_{\ell_2} \cdots \alpha_{\ell_q}
\]

\[ \rightarrow \quad \beta_k(\alpha) = \Psi_{k,p}(\alpha) + \sum_{q=0}^{p-2} (q + 1) \gamma_{q+1} \Psi_{k,q}(\alpha). \]
Harmonic Balance equations turn into system of polynomial equations

E.g. for $\psi(y) = y^3$ and a signal $y$ with 3 harmonics

\[ 0 = \alpha_0 - H(0)\beta_0(\alpha) = \]
\[ \alpha_0 - H(0) \cdot \left( \alpha_0^3 + 6\alpha_1\alpha_{-1}\alpha_0 + 3\alpha_1^2\alpha_{-2} + 3\alpha_2\alpha_{-1}^2 + 
 6\alpha_2\alpha_0\alpha_{-2} + 6\alpha_2\alpha_1\alpha_{-3} + 6\alpha_3\alpha_{-1}\alpha_{-2} + 6\alpha_3\alpha_0\alpha_{-3} \right) \]

\[ 0 = \alpha_1 - H(j\omega)\beta_1(\alpha) = \]
\[ \alpha_1 - H(j\omega) \cdot \left( 3\alpha_1\alpha_0^2 + 3\alpha_1^2\alpha_{-1} + 6\alpha_2\alpha_0\alpha_{-1} + 
 6\alpha_2\alpha_1\alpha_{-2} + 3\alpha_2^2\alpha_{-3} + 3\alpha_3\alpha_2^2 + 6\alpha_3\alpha_0\alpha_{-2} + 6\alpha_3\alpha_1\alpha_{-3} \right) \]

\[ \vdots \]

Unleash PHCpack on it
Double Integrator

We have a double integrator \( \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \) with a cost

\[
J = \frac{1}{T} \int_0^T \frac{x_1^2 - x_2^2}{2} + \frac{x_2^4}{4} + \frac{u^2}{10} \, dt
\]

so we get

\[
H(s) = \frac{10s^2}{s^4 + 10s^2 + 10} \quad \text{and} \quad H(j\omega) = \frac{-10\omega^2}{\omega^4 - 10\omega^2 + 1}.
\]

Note: The \( x_2 \) dependent term in the cost has “sweet spots” at \(|x_2| = 1\)

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The solutions of the Harmonic Balance equations (as functions of the fundamental frequency $\omega$) correspond to the Fourier coefficients of the optimal trajectories.
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The cost corresponding to the obtained trajectories is below the optimal steady-state cost (which is $J = 0$).
The optimal trajectories for one (red) and three (black) harmonics move slowly in the “sweet spots” of the cost function.
We consider the infinite-dimensional model of a string with one fixed end and one loose end, whose velocity is controlled:

\[
\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 h}{\partial z^2} - D \frac{\partial h}{\partial t} \\
h(L, t) \equiv 0 \\
\frac{\partial h}{\partial t}(0, t) = u(t).
\]

\(h\) denotes the vertical displacement, \(D\) is a damping coefficient.

As 1\textsuperscript{st} state and output we define the velocity at \(L/2\) and as the 2\textsuperscript{nd} state we pick the vertical displacement at the same location. We get

\[
u \mapsto \dot{h}(L/2) : G_1(s) = \frac{1}{2 \cosh(L \sqrt{s^2 + Ds/2})} \\
u \mapsto h(L/2) : G_2(s) = \frac{1}{2s \cosh(L \sqrt{s^2 + Ds/2})}.
\]
We assume that we want the string to have some average velocity, keeping the displacement at bay and minimizing control effort:

\[ J(x(0), u, T) = \frac{1}{T} \int_0^T \frac{x_2^2 - x_1^2}{2} + \frac{x_1^4}{4} + \frac{u^2}{2} \, dt \]
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This time, \( H(s) \) takes the form

\[ H(s) = \frac{-s^2}{\left(4 \cosh\left(\frac{L\sqrt{s^2+Ds}}{2}\right) \cosh\left(\frac{L\sqrt{s^2-Ds}}{2}\right) - 1\right) s^2 - 1} \]

and has infinitely many intervals where \( H(j\omega) > 0 \).
Examples Damped Wave Equation

\((G_1)_{dB}\)

\((H)_{dB}\)

\(\sim H \text{ does not roll off!}\)
The shapes look very similar, the optimal cost function however decreases as the frequency increases.
Optimal trajectory of the String
Further Reading

**Optimal Periodic Control**


**Harmonic Balance**


**Homotopy Continuation**

The dots correspond to evenly spaced time intervals. The optimal trajectory (black) spends a lot of time in the dips, whereas the two other stationary solutions spend less time there.