Combinatorics Summary

Department of Computer Science
University of California, Santa Barbara

Fall 2006

The Product Rule

If a procedure has 2 steps and there are n_1 ways to do the 1^{st} task and, *for each of these ways*, there are n_2 ways to do the 2^{nd} task, then there are $n_1 n_2$ ways to do the procedure.

The Sum Rule

If sets A and B are *disjoint*, then $|A \cup B| = |A| + |B|$.

Permutations

1. A *permutation* of a set of objects is an arrangement of these objects.
2. An arrangement of r elements of a set is called an *r-permutation*.
3. If $n \in \mathbf{Z}^+$ and $r \in \mathbf{Z}^+$ with $1 \le r \le n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = n!/(n-r)!$$

   r-permutations of a set with n elements.

Combinations

1. An *r-combination* of elements of a set is a subset with r elements.
2. The number of r-combinations (or r-subsets) of a set of n elements is denoted $C(n, r)$ or $\binom{n}{r}$. These numbers are referred to as *binomial coefficients*.
3. The number of *r-permutations* from a set of n elements, $P(n, r)$, can be counted using the product rule:
   (a) Select the r elements to be permuted from the set of n elements: $\binom{n}{r}$
   (b) Permute the r elements: $r!$

That is,

$$P(n,r) = \left(\begin{array}{c} n \\ r \end{array} \right) P(r,r).$$

Thus,

$$\left(\begin{array}{c} n \\ r \end{array} \right) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!} = n(n-1)(n-2)\cdots(n-r+1)/r!.$$

4. For every subset of r elements, A, there is a corresponding subset, $\overline{A}$, of $n-r$ elements: The number of r-subsets equals the number of $(n-r)$-subsets:

$$\left(\begin{array}{c} n \\ r \end{array} \right) = \left(\begin{array}{c} n \\ n-r \end{array} \right).$$

The Binomial Theorem

$$\begin{aligned} (x+y)^n &= \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) x^{n-j} y^j \\ &= \left(\begin{array}{c} n \\ 0 \end{array} \right) x^n y^0 + \left(\begin{array}{c} n \\ 1 \end{array} \right) x^{n-1} y^1 + \left(\begin{array}{c} n \\ 2 \end{array} \right) x^{n-2} y^2 + \cdots + \left(\begin{array}{c} n \\ n \end{array} \right) x^0 y^n. \end{aligned}$$

1. Evaluating the Binomial Theorem at $x = y = 1$, we get

$$\begin{aligned} 2^n &= \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) \\ &= \left(\begin{array}{c} n \\ 0 \end{array} \right) + \left(\begin{array}{c} n \\ 1 \end{array} \right) + \left(\begin{array}{c} n \\ 2 \end{array} \right) + \cdots + \left(\begin{array}{c} n \\ n \end{array} \right). \end{aligned}$$

2. Evaluating the Binomial Theorem at $x = 1$ and $y = -1$, we get

$$0 = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) (-1)^j$$

Moving all the negative terms to the other side, we get

$$\left(\begin{array}{c} n \\ 0 \end{array} \right) + \left(\begin{array}{c} n \\ 2 \end{array} \right) + \left(\begin{array}{c} n \\ 4 \end{array} \right) + \cdots = \left(\begin{array}{c} n \\ 1 \end{array} \right) + \left(\begin{array}{c} n \\ 3 \end{array} \right) + \left(\begin{array}{c} n \\ 5 \end{array} \right) + \cdots.$$

3. Any valid manipulation of the Binomial Theorem yields some identity involving binomial coefficients.

Some other Binomial Identities

We can use committee arguments to arrive at other binomial identities.

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Combinations with Repetition

- How many ways are there to select n items from a set of r elements *when repetition is allowed*?
- How many *nonnegative integer* solutions are there to the equation

$$x_1 + x_2 + \cdots + x_r = n?$$

- How many ways are there to distribute n identical objects into r distinct boxes?

The answer to the questions above is $\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$.

Arrangements with Repetition

- How many arrangements are there of n_1 objects of type 1, n_2 objects of type 2, …, n_r objects of type r, where $n_1 + n_2 + \cdots + n_r = n$?
- How many ways are there to distribute n distinct objects into r distinct boxes so that n_i objects are put in box i, for $i = 1, 2, \ldots, r$?

The answer to the questions above is

$$\frac{n!}{n_1!n_2!\cdots n_r!}.$$

Inclusion-Exclusion

Let's say that we want to count a set of objects that can be characterized as having *either* property $P_1, P_2, \ldots,$ *or* P_n. Further, A_i is the set of objects that have property P_i, for $i = 1, 2, \ldots, n$. Then the set of objects we are interested in counting is

$$A_1 \cup A_2 \cup \cdots \cup A_n.$$

The inclusion-exclusion formula allows us to count the elements in this union of sets, even though the sets may not be disjoint:

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \\
&+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}
$$

Now, let's say that we want to count a set of objects that can be characterized as having property $P_1, P_2, \ldots,$ *and* P_n. Further, A_i is the set of objects that *do not have* property P_i, for $i = 1, 2, \ldots, n$. Then, the set of objects we are interested in counting is

$$\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

However, by De Morgan's law,

$$\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} = \overline{A_1 \cup A_2 \cup \cdots \cup A_n} = U - (A_1 \cup A_2 \cup \cdots \cup A_n).$$

The inclusion-exclusion formula thus allows us to solve such counting problems.